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A. S. SOLODOVNIKOV

# SYSTEMS OF LINEAR INEQUALITIES

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The book tells about the relation of systems of linear inequalities to convex polyhedra, gives a description of the set of all solutions of a system of linear inequalities, analyses the questions of compatibility and incompatibility; finally, it gives an insight into linear programming as one of the topics in the theory of systems of linear inequalities. The last section but one gives a proof of the duality theorem of linear programming.

**The book is intended for senior pupils and all amateur mathematicians.**

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ИЗДАТЕЛЬСТВО МИР

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# SYSTEMS OF LINEAR INEQUALITIES

Translated from the Russian
by
Vladimir Shokurov

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. С. Солодовников

# СИСТЕМЫ ЛИНЕЙНЫХ НЕРАВЕНСТВ

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*На английском языке*

# Preface

First-degree or, to use the generally accepted term, *linear inequalities* are inequalities of the form

$$ax + by + c \geqslant 0$$

(for simplicity we have written an inequality in two unknowns $x$ and $y$). The theory of systems of linear inequalities is a small but most fascinating branch of mathematics. Interest in it is to a considerable extent due to the beauty of geometrical content, for in geometrical terms giving a system of linear inequalities in two or three unknowns means giving a convex polygonal region in the plane or a convex polyhedral solid in space, respectively. For example, the study of convex polyhedra, a part of geometry as old as the hills, turns thereby into one of the chapters of the theory of systems of linear inequalities. This theory has also some branches which are near the algebraist's heart; for example, they include a remarkable analogy between the properties of linear inequalities and those of systems of *linear equations* (everything connected with linear equations has been studied for a long time and in much detail).

Until recently one might think that linear inequalities would forever remain an object of purely mathematical work. The situation has changed radically since the mid 40s of this century when there arose a new area of applied mathematics—*linear programming*—with important applications in the economy and engineering. Linear programming is in the end nothing but a part (though a very important one) of the theory of systems of linear inequalities.

It is exactly the aim of this small book to acquaint the reader with the various aspects of the theory of systems of linear inequalities, viz. with the geometrical aspect of the matter and some of the methods for solving systems connected with that aspect, with certain purely algebraic properties of the systems, and with questions of linear programming. Reading the book will not require any knowledge beyond the school course in mathematics.

A few words are in order about the history of the questions to be elucidated in this book.

Although by its subject-matter the theory of linear inequalities should, one would think, belong to the most basic and elementary parts of mathematics, until recently it was studied relatively little. From the last years of the last century works began occasionally to appear which elucidated some properties of systems of linear inequalities. In this connection one can mention the names of such mathematicians as H. Minkowski (one of the greatest geometers of the end of the last and the beginning of this century especially well known for his works on convex sets and as the creator of "Minkowskian geometry"), G.F. Voronoi (one of the fathers of the "Petersburg school of number theory"), A. Haar (a Hungarian mathematician who won recognition for his works on "group integration"), H. Weyl (one of the most outstanding mathematicians of the first half of this century; one can read about his life and work in the pamphlet "Herman Weyl" by I. M. Yaglom, Moscow, "Znanie", 1967). Some of the results obtained by them are to some extent or other reflected in the present book (though without mentioning the authors' names).

It was not until the 1940s or 1950s, when the rapid growth of applied disciplines (linear, convex and other modifications of "mathematical programming", the so-called "theory of games", etc.) made an advanced and systematic study of linear inequalities a necessity, that a really intensive development of the theory of systems of linear inequalities began. At present a complete list of books and papers on inequalities would probably contain hundreds of titles.

## 1. Some Facts from Analytic Geometry

1°. *Operations on points.* Consider a plane with a rectangular coordinate system. The fact that a point $M$ has coordinates $x$ and $y$ in this system is written down as follows:

$$M = (x, y) \text{ or simply } M(x, y)$$

The presence of a coordinate system allows one to perform some operations on the points of the plane, namely *the operation of addition of points and the operation of multiplication of a point by a number.*

The addition of points is defined in the following way: if $M_1 = (x_1, y_1)$ and $M_2 = (x_2, y_2)$, then

$$M_1 + M_2 = (x_1 + x_2, y_1 + y_2)$$

Thus the addition of points is reduced to the addition of their similar coordinates.

The visualization of this operation is very simple (Fig. 1); the point $M_1 + M_2$ is the fourth vertex of the parallelogram constructed on the segments $OM_1$ and $OM_2$ as its sides ($O$ is the origin of coordinates). $M_1$, $O$, $M_2$ are the three remaining vertices of the parallelogram.

The same can be said in another way: the point $M_1 + M_2$ is obtained by translating the point $M_2$ in the direction of the segment $OM_1$ over a distance equal to the length of the segment.

The multiplication of the point $M(x, y)$ by an arbitrary number $k$ is carried out according to the following rule:

$$kM = (kx,\ ky)$$

The visualization of this operation is still simpler than that of the addition; for $k > 0$ the point $M' = kM$ lies on the ray $OM$, with

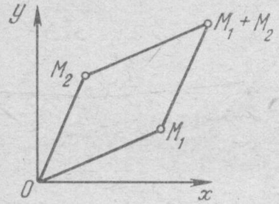


Fig. 1

$OM' = k \times OM$; for $k < 0$ the point $M'$ lies on the extension of the ray $OM$ beyond the point $O$, with $OM' = |k| \times OM$ (Fig. 2).

The derivation of the above visualization of both operations will provide a good exercise for the reader*.

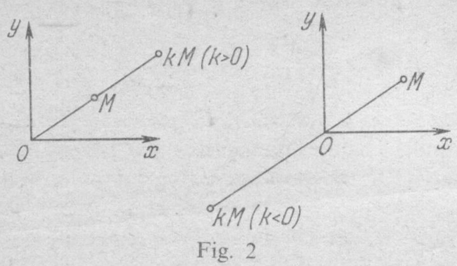


Fig. 2

The operations we have introduced are very convenient to use in interpreting geometric facts in terms of algebra. We cite some examples to show this.

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* Unless the reader is familiar with the fundamentals of vector theory. In vector terms our operations are known to mean the following: the point $M_1 + M_2$ is the end of the vector $\vec{OM}_1 + \vec{OM}_2$ and the point $kM$ is the end of the vector $k \times \vec{OM}$ (on condition that the point $O$ is the beginning of this vector).

(1) *The segment* $M_1M_2$ *consists of all points of the form*

$$s_1M_1 + s_2M_2$$

*where* $s_1$, $s_2$ *are any two nonnegative numbers the sum of which equals* 1.

Here a purely geometric fact, the belonging of a point to the segment $M_1M_2$, is written in the form of the algebraic relation $M = s_1M_1 + s_2M_2$ with the above constraints on $s_1$, $s_2$.

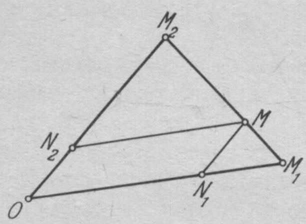


Fig. 3

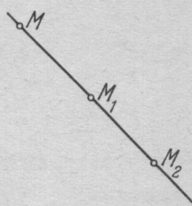


Fig. 4

To prove the above, consider an arbitrary point $M$ on the segment $M_1M_2$. Drawing through $M$ straight lines parallel to $OM_2$ and $OM_1$ we obtain the point $N_1$ on the segment $OM_1$ and the point $N_2$ on the segment $OM_2$ (Fig. 3). Let

$$s_1 = \frac{M_2M}{M_2M_1}, \quad s_2 = \frac{M_1M}{M_1M_2}$$

the numbers $s_1$ and $s_2$ being nonnegative and their sum equalling 1. From the similarity of the corresponding triangles we find

$$\frac{ON_1}{OM_1} = \frac{M_2M}{M_2M_1} = s_1, \quad \frac{ON_2}{OM_2} = \frac{M_1M}{M_1M_2} = s_2$$

which yields $N_1 = s_1M_1$, $N_2 = s_2M_2$. But $M = N_1 + N_2$, hence $M = s_1M_1 + s_2M_2$. We, finally, remark that when the point $M$ runs along the segment $M_1M_2$ in the direction from $M_1$ toward $M_2$, the number $s_2$ runs through all the values from 0 to 1. Thus proposition (1) is proved.

(2) *Any point* $M$ *of the straight line* $M_1M_2$ *can be represented as*

$$tM_1 + (1 - t)M_2$$

*where* $t$ *is a number.*

In fact, if the point $M$ lies on the segment $M_1M_2$, then our statement follows from that proved above. Let $M$ lie outside of the segment $M_1M_2$. Then either the point $M_1$ lies on the segment $MM_2$ (as in Fig. 4) or $M_2$ lies on the segment $MM_1$. Suppose, for example, that the former is the case. Then, from what has been proved,

$$M_1 = sM + (1 - s)M_2 \quad (0 < s < 1)$$

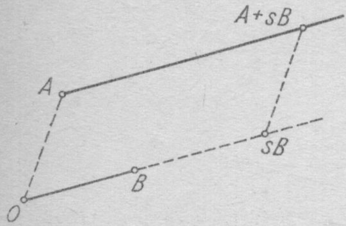


Fig. 5

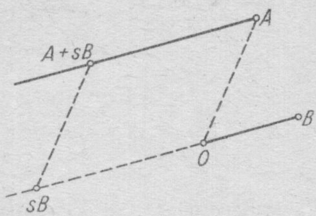


Fig. 6

Hence

$$M = \frac{1}{s} M_1 - \frac{1-s}{s} M_2 = tM_1 + (1 - t) M_2$$

where $t = 1/s$. Let the case where $M_2$ lies on the segment $MM_1$ be considered by the reader.

(3) *When a parameter $s$ increases from 0 to $\infty$, the point $sB$ runs along the ray $OB$* and the point $A + sB$ is the ray emerging from $A$ in the direction of $OB$. When $s$ decreases from 0 to $-\infty$, the points $sB$ and $A + sB$ run along the rays that are supplementary to those indicated above.* To establish this, it is sufficient to look at Figs. 5 and 6.

It follows from proposition (3) that, as $s$ changes from $-\infty$ to $+\infty$, the point $A + sB$ runs along the straight line passing through $A$ and parallel to $OB$.

The operations of addition and multiplication by a number can, of course, be performed on points in space as well. In that case,

* The point $B$ is supposed to be different from the origin of coordinates $O$.

by definition,

$$M_1 + M_2 = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$$
$$kM = (kx,\ ky,\ kz)$$

All the propositions proved above will obviously be true for space as well.

We conclude this section by adopting a convention which will later help us formulate many facts more clearly and laconically. Namely, if $\mathscr{K}$ and $\mathscr{L}$ are some two sets of points (in the plane or in space), then we shall agree to understand by their "sum" $\mathscr{K} + \mathscr{L}$ a set of all points of the form $K + L$ where $K$ is an arbitrary point in $\mathscr{K}$ and $L$ an arbitrary point in $\mathscr{L}$.

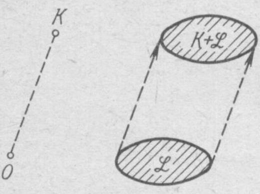

Fig. 7

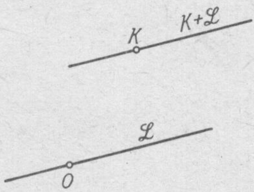

Fig. 8

Special notation has been employed in mathematics for a long time to denote the belonging of a point to a given set; namely, in order to indicate that a point $M$ belongs to a set $\mathscr{M}$ one writes $M \in \mathscr{M}$ (the symbol $\in$ standing for the word "belongs"). So $\mathscr{K} + \mathscr{L}$ is a set of all points of the form $K + L$ where $K \in \mathscr{K}$ and $L \in \mathscr{L}$.

From the visualization of the addition of points a simple rule for the addition of the point sets $\mathscr{K}$ and $\mathscr{L}$ can be given. This rule is as follows. For each point $K \in \mathscr{K}$ a set must be constructed which is a result of translating $\mathscr{L}$ along the segment $OK$ over a distance equal to the length of the segment and then all sets obtained in this way must be united into one. It is the latter that will be $\mathscr{K} + \mathscr{L}$.

We shall cite some examples.

1. Let a set $\mathscr{K}$ consist of a single point $K$ whereas $\mathscr{L}$ is any set of points. The set $K + \mathscr{L}$ is a result of translating the set $\mathscr{L}$ along the segment $OK$ over a distance equal to its length (Fig. 7). In particular, if $\mathscr{L}$ is a straight line, then $K + \mathscr{L}$ is a straight line parallel to $\mathscr{L}$. If at the same time the line $\mathscr{L}$ passes through the origin, then $K + \mathscr{L}$ is a straight line parallel to $\mathscr{L}$ and passing through the point $K$ (Fig. 8).

2. $\mathscr{K}$ and $\mathscr{L}$ are segments (in the plane or in space) not parallel to each other (Fig. 9). Then the set $\mathscr{K}+\mathscr{L}$ is a parallelogram with sides equal and parallel to $\mathscr{K}$ and $\mathscr{L}$ (respectively). What will result if the segments $\mathscr{K}$ and $\mathscr{L}$ are parallel?

3. $\mathscr{K}$ is a plane and $\mathscr{L}$ is a segment not parallel to it. The set $\mathscr{K}+\mathscr{L}$ is a part of space lying between two planes parallel to $\mathscr{K}$ (Fig. 10).

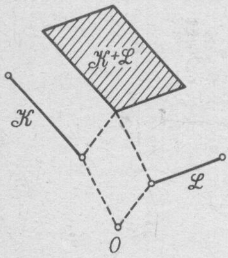

Fig. 9

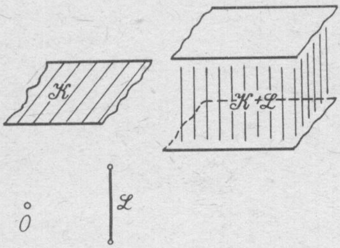

Fig. 10

4. $\mathscr{K}$ and $\mathscr{L}$ are circles of radii $r_1$ and $r_2$ with centres $P_1$ and $P_2$ (respectively) lying in the same plane $\pi$. Then $\mathscr{K}+\mathscr{L}$ is a circle of radius $r_1+r_2$ with the centre at the point $P_1+P_2$ lying in a plane parallel to $\pi$ (Fig. 11).

2°. *The visualization of equations and inequalities of the first degree in two or three unknowns.* Consider a first-degree equation in two unknowns $x$ and $y$:

$$ax+by+c=0 \tag{1}$$

Interpreting $x$ and $y$ as coordinates of a point in the plane, it is natural to ask the question: What set is formed in the plane by the points whose coordinates satisfy equation (1), or in short what set of points is given by equation (1)?

We shall give the answer though the reader may already know it: *the set of points given by equation* (1) *is a straight line in the plane.* Indeed, if $b \neq 0$, then equation (1) is reduced to the form

$$y=kx+p$$

and this equation is known to give a straight line. If, however, $b=0$, then the equation is reduced to the form

$$x=h$$

and gives a straight line parallel to the axis of ordinates.

A similar question arises concerning the inequality

$$ax + by + c \geqslant 0 \tag{2}$$

What set of points in the plane is given by inequality (2)?

Here again the answer is very simple. If $b \neq 0$, then the inequality is reduced to one of the following forms

$$y \geqslant kx + p \quad \text{or} \quad y \leqslant kx + p$$

It is easy to see that the first of these inequalities is satisfied by

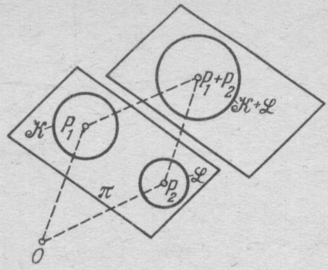


Fig. 11

all points lying "above" or on the straight line $y = kx + p$ and the second by all points lying "below" or on the line (Fig. 12). If, however, $b = 0$, then the inequality is reduced to one of the following forms

$$x \geqslant h \quad \text{or} \quad x \leqslant h$$

the first of them being satisfied by all points lying to the "right" of or on the straight line $x = h$ and the second by all points to the "left" of or on the line (Fig. 13).

Thus *equation* (1) *gives a straight line in the coordinate plane and inequality* (2) *gives one of the two half-planes into which this line divides the whole plane* (the line itself is considered to belong to either of these two half-planes).

We now want to solve similar questions with regard to the equation

$$ax + by + cz + d = 0 \tag{3}$$

and the inequality

$$ax + by + cz + d \geqslant 0 \tag{4}$$

of course, here $x$, $y$, $z$ are interpreted as coordinates of a point *in space*. It is not difficult to foresee that the following result will be obtained.

Theorem. *Equation* (3) *gives a plane in space and inequality* (4) *gives one of the two half-spaces into which this plane divides the whole space* (the plane itself is considered to belong to one of these two half-spaces).

*Proof*. Of the three numbers $a$, $b$, $c$ at least one is different

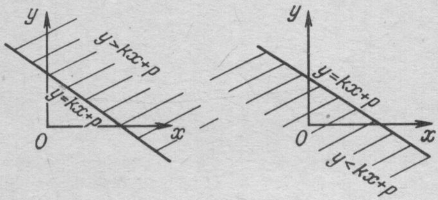


Fig. 12

from zero; let $c \neq 0$, for example. Then equation (3) is reduced to the form

$$z = kx + ly + p \tag{5}$$

Denote by $\mathscr{L}$ the set of all points $M$ $(x, y, z)$ which satisfy (5). Our aim is to show that $\mathscr{L}$ is a plane.

Find what points in $\mathscr{L}$ belong to the $yOz$ coordinate plane. To do this, set $x = 0$ in (5) to obtain

$$z = ly + p \tag{6}$$

Thus the intersection of $\mathscr{L}$ with the $yOz$ plane is the straight line $u$ given in the plane by equation (6) (Fig. 14).

Similarly, we shall find that the intersection of $\mathscr{L}$ with the $xOz$ plane is the straight line $v$ given in the plane by the equation

$$z = kx + p \tag{7}$$

Both lines $u$ and $v$ pass through the point $P(0, 0, p)$.

Denote by $\pi$ the plane containing the lines $u$ and $v$. Show that $\pi$ *belongs* to the set $\mathscr{L}$.

In order to do this it is sufficient to establish the following fact, viz. that a straight line passing through any point $A \in v$ and parallel to $u$ belongs to $\mathscr{L}$.

First find a point $B$ such that $OB \| u$. The equation $z = ly + p$ gives the straight line $u$ in the $yOz$ plane; hence the equation $z = ly$ gives a straight line parallel to $u$ and passing through the origin (it is shown as dotted line in Fig. 14). We can take as $B$ the point with the coordinates $y=1$, $z=l$ which lies on this line.

An arbitrary point $A \in v$ has the coordinates $x$, 0, $kx+p$. The point $B$ we have chosen has the coordinates 0, 1, $l$. The straight line passing through $A$ and parallel to $u$ consists of the points

$$A + sB = (x,\ 0,\ kx+p) + s(0,\ 1,\ l) =$$
$$= (x,\ s,\ kx+p+sl)$$

where $s$ is an arbitrary number (see proposition (3) of section 1°).

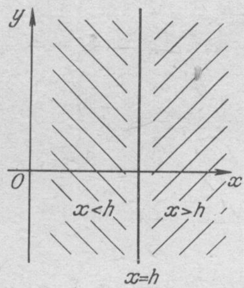


Fig. 13

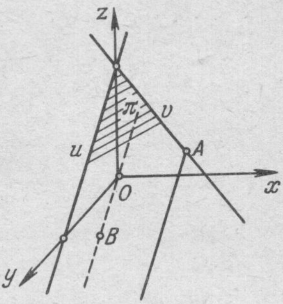


Fig. 14

It is easy to check that the coordinates of a point $A+sB$ satisfy equation (5), i.e. that $A+sB \in \mathscr{L}$. This proves that the plane $\pi$ belongs wholly to the set $\mathscr{L}$.

It remains to make the last step, to show that $\mathscr{L}$ *coincides* with $\pi$ or, in other words, that the set $\mathscr{L}$ does not contain any points outside $\pi$.

To do this, consider three points: a point $M$ $(x_0,\ y_0,\ z_0)$ lying in the plane $\pi$, a point $M'$ $(x_0,\ y_0,\ z_0+\varepsilon)$ lying "above" the plane $\pi$ $(\varepsilon > 0)$, and a point $M''$ $(x_0,\ y_0,\ z_0-\varepsilon)$ lying "below" $\pi$ (Fig. 15). Since $M \in \pi$, we have $z_0 = kx_0 + ly_0 + p$ and hence

$$z_0 + \varepsilon > kx_0 + ly_0 + p$$
$$z_0 - \varepsilon < kx_0 + ly_0 + p$$

This shows that the coordinates of the point $M'$ satisfy the strict inequality

$$z > kx + ly + p$$

and the coordinates of the point $M''$ satisfy the strict inequality

$$z < kx + ly + p$$