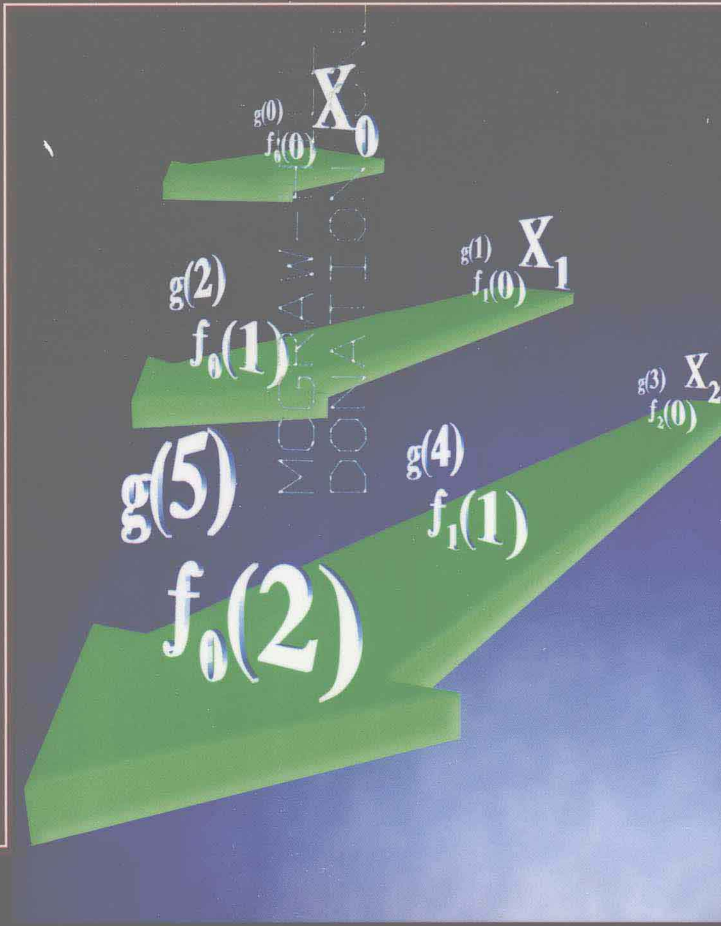
The Walter Rudin Student Series in Advanced Mathematics

Bob A. Dumas
John E. McCarthy

Transition to Higher Mathematics

Structure and Proof



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To Gloria, Siena, and William
B. D.

To Suzanne, Fiona, and Myles
J. M^{c}C.

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CHAPTER 0

Introduction

0.1 Why This Book Is

More students today than ever before take calculus in high school. This comes at a cost, however: fewer and fewer take a rigorous course in Euclidean geometry. Moreover, the calculus course taken by almost all students, whether in high school or college, avoids proofs and often does not even give a formal definition of a limit. Indeed some students enter the university having never read or written a proof by induction or encountered a mathematical proof of any kind.

As a consequence, teachers of upper-level undergraduate mathematics courses in linear algebra, abstract algebra, analysis, and topology have to work extremely hard inculcating the concept of proof while simultaneously trying to cover the syllabus. This problem has been addressed at many colleges and universities by introducing a bridge course, with a title like "Foundations for Higher Mathematics," taken by students who have completed the regular calculus sequence. Some of these students plan to become mathematics majors. Others just want to learn some more mathematics; but if what they are exposed to is interesting and satisfying, many will choose to major or double major in mathematics.

This book is written for students who have taken calculus and want to learn what "real mathematics" is. We hope you will find the material engaging and interesting, and that you will be encouraged to learn more advanced mathematics.

0.2 What This Book Is

The purpose of this book is to introduce you to the culture, language, and thinking of mathematicians. We say "mathematicians," not "mathematics," to emphasize that mathematics is, at heart, a human endeavor. If there is intelligent life in Erewhemos, then the Erewhemosians will surely agree that two plus two equals four. If they have thought carefully about the question, they will not believe that the square root of two can be exactly given by the ratio of two whole numbers or that there are finitely many prime numbers. However, we can only speculate about whether they would find these latter questions remotely interesting or what they might consider satisfying answers to questions of this kind.

Mathematicians have, after millennia of struggles and arguments, reached a widespread (if not quite universal) agreement as to what constitutes an acceptable mathematical argument. They call this a "proof," and it constitutes a carefully reasoned argument based on agreed premises. The methodology of mathematics has been spectacularly successful, and it has spawned many other fields. In the twentieth century, computer programming and applied statistics developed from offshoots of mathematics into disciplines of their own. In the nineteenth century, so did astronomy and physics. The increasing availability of data make the treatment of data in a sophisticated mathematical way one of the great scientific challenges of the twenty-first century.

In this book, we shall try to teach you what a proof is—what level of argument is considered convincing, what is considered over-reaching, and what level of detail is considered too much. We shall try to teach you how mathematicians think—what structures they use to organize their thoughts. A structure is like a skeleton—if you strip away the inessential details you can focus on the real problem. A great example of this is the idea of number, the earliest human mathematical structure. If you learn how to count apples and that two apples plus two apples make four apples, and if you think that this is about apples rather than counting, then you still do not know what two sheep plus two sheep make. But once you realize that there is an underlying structure of number and that two plus two is four in the abstract, then adding wool or legs to the objects does not change the arithmetic.

0.3 What This Book Is Not

There is an approach to teaching a transition course that many instructors favor. It is to have a problem-solving course in which students learn to write proofs in a context where their intuition can help, such as in combinatorics or number theory. This helps to make the course interesting and can keep students from getting totally lost.

We have not adopted this approach. Our reason is that in addition to teaching the skill of writing a logical proof, we also want to teach the skill of carefully analyzing definitions. Much of the instructor's labor in an upper-division algebra or analysis course consists of forcing the students to carefully read the definitions of new and unfamiliar objects, to decide which mathematical objects satisfy the definition and which do not, and to understand what follows "immediately" from the definitions. Indeed, the major reason that the epsilon-delta definition of limit has disappeared from most introductory calculus courses is the difficulty of explaining how the quantifiers $\forall\varepsilon\ \exists\delta$, in precisely this order, give the exact notion of limit for which we are striving. Thus, while students must work harder in this course to learn more abstract mathematics, they will be better prepared for advanced courses.

Nor is this a text in applied logic. The early chapters of the book introduce the student to the basic mathematical structures through formal definitions. Although we provide a rather formal treatment of first-order logic and mathematical induction, our objective is to move to more advanced classical *mathematical* structures and arguments as soon as the student has an adequate understanding of the logic underlying mathematical proofs.

0.4 Advice to the Student

Welcome to higher mathematics! If your exposure to college-level mathematics is limited to calculus, this book will probably seem very different from your previous texts. Many students learn calculus by quickly scanning the text and proceeding directly to the problems. When struggling with a problem, they seek similar problems in the text and attempt to emulate the solution they find. Finally, they check the solution, usually found at the back of the text, to "validate" the methodology.

This book, like many texts addressing more advanced topics, is not written with computational problems in mind. Our objective is to introduce you to the various elements of higher undergraduate mathematics—the culture, language, methods, topics, standards, and results. The problems in these courses are to prove true mathematical claims or refute untrue claims. In the context of calculus, the mathematician must prove the results that you freely used. To most people, this activity seems very different from computation. For instance, you will probably find it necessary to think about a problem for some time before you begin writing. Unlike calculus, in which the general direction of the methods is usually obvious, trying to prove mathematical claims can feel directionless or accidental. However, it is strategic rather than random. This is one of the great challenges of mathematics—at the higher levels, it is creative, not rote. With practice and disciplined thinking, you will learn to see your way to proving mathematical claims.

We shall begin our treatment of higher mathematics with a large number of definitions. This is usual in a mathematics course and is necessary because mathematics requires precise expression. We shall try to motivate these definitions so that their usefulness will be obvious as early as possible. After presenting and discussing some definitions, we shall present arguments for some elementary claims concerning these definitions. This will give us some practice in reading, writing, and discussing mathematics. In the early chapters of the book, we include numerous discussions and remarks to help you grasp the basic direction of the arguments. In the later chapters of the book, you will read more difficult arguments for some deep classical results. We recommend that you read these arguments deliberately to ensure your thorough understanding of the argument and to nurture your sense of the level of detail and rigor expected in an undergraduate mathematical proof.

There are exercises at the end of each chapter designed to direct your attention to the reading and compel you to think through the details of the proofs. Some of these exercises are straightforward, but many of them are very hard. We do not expect that every student will be able to solve every problem. However, spending an hour (or more) thinking about a difficult problem is time well spent even if you do not solve the problem: it strengthens your mathematical muscles and allows you to appreciate, and to understand more deeply, the solution if it is eventually shown to you. Ultimately, you will be able to solve some of the hard problems yourself after thinking deeply about them. Then you will be a real mathematician!

Mathematics is, from one point of view, a logical exercise. We define objects that do not physically exist and use logic to draw the deepest conclusions we can concerning these objects. If this were the end of the story, mathematics would be no more than a game and would be of little enduring interest. It happens, however, that interpreting physical objects, processes, behaviors, and other subjects of intellectual interest as mathematical objects, and applying the conclusions and techniques from the study of these mathematical objects, allows us to draw reliable and powerful conclusions about practical problems. This method of using mathematics to understand the world is called mathematical modelling. The world in which you live, the way you understand this world, and how it differs from the world and understanding of your distant ancestors is to a large extent the result of mathematical investigation. In this book, we try to explain how to draw mathematical conclusions with certainty. When you studied calculus, you used numerous deep theorems in order to draw conclusions that otherwise might have taken months rather than minutes. Now we shall develop an understanding of how results of this depth and power are derived.

0.5 Advice to the Instructor

Learning terminology—what do "contrapositive" and "converse" mean—comes easily to most students. Your challenge in the course is to teach them how to read definitions closely and then how to manipulate them. This is much harder when there is no concrete image that students can keep in mind. Vectors in $\mathbb{R}^n$, for example, are more intimidating than in $\mathbb{R}^3$, not because of any great inherent increase in complexity but because they are harder to think of geometrically, so students must trust the algebra alone. This trust takes time to build.

Chapter 1 is mainly to establish notation and discuss necessary concepts that some may have already seen (like injections and surjections). Unfortunately this may be the first exposure to some of these ideas for many students, so the treatment is rather lengthy. The speed at which the material is covered naturally will depend on the strength and background of the students. Take some time explaining why a sequence can be thought of as a function with domain $\mathbb{N}$—variations on this idea will recur.

Chapter 2 introduces relations. These are hard to grasp because of the abstract nature of the definition. Equivalences and linear orderings recur throughout the book, and students' comfort with these will increase.

Neither Chapter 1 nor Chapter 2 dwell on proofs. In fact mathematical proofs and elementary first-order logic are not introduced until Chapter 3. Our objective is to get students thinking about mathematical structures and definitions without the additional psychic weight of reading and writing proofs. We use examples to illustrate the definitions. The first chapters provide basic conceptual foundations for later chapters, and we find that most students have their hands full just trying to read and understand the definitions and examples. In the exercises we ask the students to "show" the truth of some mathematical claims. Our intention is to get the student thinking about the task of proving mathematical claims. It is not expected that they will write successful arguments before Chapter 3. We encourage the students to attempt the problems even though they will likely be uncertain about the requirements for a mathematical proof. If you feel strongly that mathematical proofs need to be discussed before launching into mathematical definitions, you can cover Chapter 3 first.

Chapter 3 is fairly formal, and should go quickly. Chapter 4 introduces students to the first major proof technique—induction. With practice, they can be expected to master this technique. We also introduce as an ongoing theme the study of polynomials and prove, for example, that a polynomial has no more roots than its degree.

Chapters 5, 6, and 7 are completely independent of each other. Chapter 5 treats limits and continuity, up to proving that the uniform limit of a sequence of continuous functions is continuous. Chapter 6 is on infinite sets, proving Cantor's theorems and the Schröder-Bernstein theorem. By the end of the chapter, the students will have come to appreciate that it is generally much easier to construct two injections than one bijection!

Chapter 7 contains a little number theory—up to the proof of Fermat's little theorem. It then shows how much of the structure transfers to the algebra of real polynomials.

Chapter 8 constructs the real numbers, using Dedekind cuts, and proves that they have the least upper bound property. This is then used to prove the basic theorems of real analysis—the Intermediate Value theorem and the Extreme Value theorem. Sections 8.1–8.4 require only Chapters 1–4 and Section 6.1. Sections 8.5–8.8 require Sections 5.1 and 5.2. Section 8.9 requires Chapter 6.

In Chapter 9, we introduce the complex numbers. Sections 9.1–9.3 prove the Tartaglia-Cardano formula for finding the roots of a cubic and point out how it is necessary to use complex numbers even to find real roots of real cubics. These sections require only Chapters 1–4. In Section 9.4 we prove the Fundamental Theorem of Algebra. This requires Chapter 5 and the Bolzano-Weierstrass theorem from Section 8.6.

What is a reasonable course based on this book? Chapters 1–4 are essential for any course. In a one-quarter course, one could also cover Chapter 6 and either Chapter 5 or 7. In a semester-long course, one could cover Chapters 1–6 and one of the remaining three chapters. Chapter 9 can be covered without Chapter 8 if one is willing to assert the least upper bound property as an axiom of the real numbers, and then Section 8.6 can be covered before Section 9.4 without any other material from Chapter 8.

We suggest that you agree with your colleagues on a common curriculum for this course so that topics that you cover thoroughly (e.g., cardinality) need not be repeated in successive courses.