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An Introduction to Algebraic Topology

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Joseph J. Rotman

An Introduction to Algebraic Topology

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Joseph J. Rotman

An Introduction to Algebraic Topology

With 92 Illustrations

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To my wife Marganit
and my children Ella Rose and Daniel Adam
without whom this book would have
been completed two years earlier

Preface

There is a canard that every textbook of algebraic topology either ends with the definition of the Klein bottle or is a personal communication to J. H. C. Whitehead. Of course, this is false, as a glance at the books of Hilton and Wylie, Maunder, Munkres, and Schubert reveals. Still, the canard does reflect some truth. Too often one finds too much generality and too little attention to details.

There are two types of obstacle for the student learning algebraic topology. The first is the formidable array of new techniques (e.g., most students know very little homological algebra); the second obstacle is that the basic definitions have been so abstracted that their geometric or analytic origins have been obscured. I have tried to overcome these barriers. In the first instance, new definitions are introduced only when needed (e.g., homology with coefficients and cohomology are deferred until after the Eilenberg–Steenrod axioms have been verified for the three homology theories we treat—singular, simplicial, and cellular). Moreover, many exercises are given to help the reader assimilate material. In the second instance, important definitions are often accompanied by an informal discussion describing their origins (e.g., winding numbers are discussed before computing $\pi_1(S^1)$, Green's theorem occurs before defining homology, and differential forms appear before introducing cohomology).

We assume that the reader has had a first course in point-set topology, but we do discuss quotient spaces, path connectedness, and function spaces. We assume that the reader is familiar with groups and rings, but we do discuss free abelian groups, free groups, exact sequences, tensor products (always over **Z**), categories, and functors.

I am an algebraist with an interest in topology. The basic outline of this book corresponds to the syllabus of a first-year's course in algebraic topology designed by geometers and topologists at the University of Illinois, Urbana;. other expert advice came (indirectly) from my teachers, E. H. Spanier and S. Mac Lane, and from J. F. Adam's *Algebraic Topology: A Student's Guide*. This latter book is strongly recommended to the reader who, having finished this book, wants direction for further study.

I am indebted to the many authors of books on algebraic topology, with a special bow to Spanier's now classic text. My colleagues in Urbana, especially Ph. Tondeur, H. Osborn, and R. L. Bishop, listened and explained. M.-E. Hamstrom took a particular interest in this book; she read almost the entire manuscript and made many wise comments and suggestions that have improved the text; my warmest thanks to her. Finally, I thank Mrs. Dee Wrather for a superb job of typing and Springer-Verlag for its patience.

Joseph J. Rotman

To the Reader

Doing exercises is an essential part of learning mathematics, and the serious reader of this book should attempt to solve all the exercises as they arise. An asterisk indicates only that an exercise is cited elsewhere in the text, sometimes in a proof (those exercises used in proofs, however, are always routine).

I have never found references of the form 1.2.1.1 convenient (after all, one decimal point suffices for the usual description of real numbers). Thus, Theorem 7.28 here means the 28th theorem in Chapter 7.

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CHAPTER 0

Introduction

One expects algebraic topology to be a mixture of algebra and topology, and that is exactly what it is. The fundamental idea is to convert problems about topological spaces and continuous functions into problems about algebraic objects (e.g., groups, rings, vector spaces) and their homomorphisms; the method may succeed when the algebraic problem is easier than the original one. Before giving the appropriate setting, we illustrate how the method works.

Notation

Let us first introduce notation for some standard spaces that is used throughout the book.

$\mathbf{Z}$ = integers (positive, negative, and zero).
$\mathbf{Q}$ = rational numbers.
$\mathbf{C}$ = complex numbers.
$\mathbf{I} = [0, 1]$, the (closed) unit interval.
$\mathbf{R}$ = real numbers.
$\mathbf{R}^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbf{R} \text{ for all } i\}$.

$\mathbf{R}^n$ is called **real n-space** or **euclidean space** (of course, $\mathbf{R}^n$ is the cartesian product of n copies of $\mathbf{R}$). Also, $\mathbf{R}^2$ is homeomorphic to $\mathbf{C}$; in symbols, $\mathbf{R}^2 \approx \mathbf{C}$. If $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, then its **norm** is defined by $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ (when $n = 1$, then $\|x\| = |x|$, the absolute value of x). We regard $\mathbf{R}^n$ as the subspace of $\mathbf{R}^{n+1}$ consisting of all $(n+1)$-tuples having last coordinate zero.

$$S^n = \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}.$$

S^n is called the **n-sphere** (of radius 1 and center the origin). Observe that $S^n \subset \mathbf{R}^{n+1}$(as the circle $S^1 \subset \mathbf{R}^2$); note also that the 0-sphere S^0 consists of the two points $\{1, -1\}$ and hence is a discrete two-point space. We may regard S^n as the **equator** of S^{n+1}:

$$S^n = \mathbf{R}^{n+1} \cap S^{n+1} = \{(x_1, \ldots, x_{n+2}) \in S^{n+1} : x_{n+2} = 0\}.$$

The **north pole** is $(0, 0, \ldots, 0, 1) \in S^n$; the **south pole** is $(0, 0, \ldots, 0, -1)$. The **antipode** of $x = (x_1, \ldots, x_{n+1}) \in S^n$ is the other endpoint of the diameter having one endpoint x; thus the antipode of x is $-x = (-x_1, \ldots, -x_{n+1})$, for the distance from $-x$ to x is 2.

$$D^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}.$$

D^n is called the **n-disk** (or **n-ball**). Observe that $S^{n-1} \subset D^n \subset \mathbf{R}^n$; indeed S^{n-1} is the boundary of D^n in $\mathbf{R}^n$.

$$\Delta^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : \text{each } x_i \geq 0 \text{ and } \textstyle\sum x_i = 1\}.$$

Δ^n is called the **standard n-simplex**. Observe that Δ^0 is a point, Δ^1 is a closed interval, Δ^2 is a triangle (with interior), Δ^3 is a (solid) tetrahedron, and so on. It is obvious that $\Delta^n \approx D^n$, although the reader may not want to construct[1] a homeomorphism until Exercise 2.11.

There is a standard homeomorphism from $S^n - \{\text{north pole}\}$ to $\mathbf{R}^n$, called **stereographic projection**. Denote the north pole by N, and define $\sigma: S^n - \{N\} \to \mathbf{R}^n$ to be the intersection of $\mathbf{R}^n$ and the line joining x and N. Points on the latter line have the form $tx + (1-t)N$; hence they have coordinates $(tx_1, \ldots, tx_n, tx_{n+1} + (1-t))$. The last coordinate is zero for $t = (1 - x_{n+1})^{-1}$; hence

$$\sigma(x) = (tx_1, \ldots, tx_n),$$

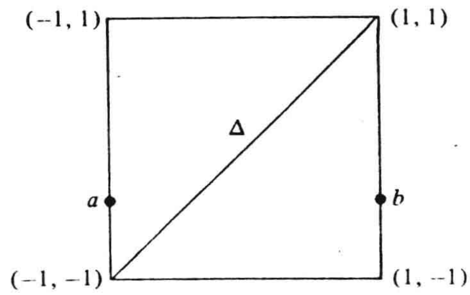
where $t = (1 - x_{n+1})^{-1}$. It is now routine to check that σ is indeed a homeomorphism. Note that $\sigma(x) = x$ if and only if x lies on the equator S^{n-1}.

Brouwer Fixed Point Theorem

Having established notation, we now sketch a proof of the **Brouwer fixed point theorem**: if $f: D^n \to D^n$ is continuous, then there exists $x \in D^n$ with $f(x) = x$. When $n = 1$, this theorem has a simple proof. The disk D^1 is the closed interval $[-1, 1]$; let us look at the graph of f inside the square $D^1 \times D^1$.

---

[1] It is an exercise that a compact convex subset of $\mathbf{R}^n$ containing an interior point is homeomorphic to D^n (convexity is defined in Chapter 1); it follows that Δ^n, D^n, and $\mathbf{I}^n$ are homeomorphic.



Theorem 0.1. *Every continuous* $f\colon D^1 \to D^1$ *has a fixed point.*

PROOF. Let $f(-1) = a$ and $f(1) = b$. If either $f(-1) = -1$ or $f(1) = 1$, we are done. Therefore, we may assume that $f(-1) = a > -1$ and that $f(1) = b < 1$, as drawn. If G is the graph of f and Δ is the graph of the identity function (of course, Δ is the diagonal), then we must prove that $G \cap \Delta \neq \varnothing$. The idea is to use a connectedness argument to show that every path in $D^1 \times D^1$ from a to b must cross Δ. Since f is continuous, $G = \{(x, f(x)) : x \in D^1\}$ is connected [G is the image of the continuous map $D^1 \to D^1 \times D^1$ given by $x \mapsto (x, f(x))$]. Define $A = \{(x, f(x)) : f(x) > x\}$ and $B = \{(x, f(x)) : f(x) < x\}$. Note that $a \in A$ and $b \in B$, so that $A \neq \varnothing$ and $B \neq \varnothing$. If $G \cap \Delta = \varnothing$, then G is the disjoint union

$$G = A \cup B.$$

Finally, it is easy to see that both A and B are open in G, and this contradicts the connectedness of G. □

Unfortunately, no one knows how to adapt this elementary topological argument when $n > 1$; some new idea must be introduced. There is a proof using the *simplicial approximation theorem* (see [Hirsch]). There are proofs by analysis (see [Dunford and Schwartz, pp. 467–470] or [Milnor (1978)]); the basic idea is to approximate a continuous function $f\colon D^n \to D^n$ by smooth functions $g\colon D^n \to D^n$ in such a way that f has a fixed point if all the g do; one can then apply analytic techniques to smooth functions.

Here is a proof of the Brouwer fixed point theorem by algebraic topology. We shall eventually prove that, for each $n \geq 0$, there is a *homology functor* H_n with the following properties: for each topological space X there is an abelian group $H_n(X)$, and for each continuous function $f\colon X \to Y$ there is a homomorphism $H_n(f)\colon H_n(X) \to H_n(Y)$, such that:

$$H_n(g \circ f) = H_n(g) \circ H_n(f) \tag{1}$$

whenever the composite $g \circ f$ is defined;

$$H_n(1_X) \text{ is the identity function on } H_n(X), \tag{2}$$

where 1_X is the identity function on X;

$$H_n(D^{n+1}) = 0 \quad \text{for all } n \geq 1; \tag{3}$$

$$H_n(S^n) \neq 0 \quad \text{for all } n \geq 1. \tag{4}$$

Using these H_n's, we now prove the Brouwer theorem.

Definition. A subspace X of a topological space Y is a **retract** of Y if there is a continuous map[2] $r: Y \to X$ with $r(x) = x$ for all $x \in X$; such a map r is called a **retraction**.

Remarks. (1) Recall that a topological space X contained in a topological space Y is a **subspace** of Y if a subset V of X is open in X if and only if $V = X \cap U$ for some open subset U of Y. Observe that this guarantees that the inclusion $i: X \hookrightarrow Y$ is continuous, because $i^{-1}(U) = X \cap U$ is open in X whenever U is open in Y. This parallels group theory: a group H contained in a group G is a **subgroup** of G if and only if the inclusion $i: H \hookrightarrow G$ is a homomorphism (this says that the group operations in H and in G coincide).

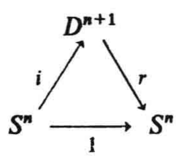
(2) One may rephrase the definition of retract in terms of functions. If $i: X \hookrightarrow Y$ is the inclusion, then a continuous map $r: Y \to X$ is a retraction if and only if

$$r \circ i = 1_X.$$

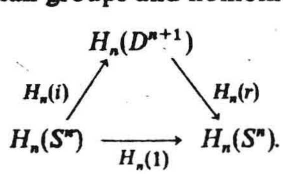
(3) For abelian groups, one can prove that a subgroup H of G is a retract of G if and only if H is a **direct summand** of G; that is, there is a subgroup K of G with $K \cap H = 0$ and $K + H = G$ (see Exercise 0.1).

Lemma 0.2. *If $n \geq 0$, then S^n is not a retract of D^{n+1}.*

PROOF. Suppose there were a retraction $r: D^{n+1} \to S^n$; then there would be a "commutative diagram" of topological spaces and continuous maps



(here commutative means that $r \circ i = 1$, the identity function on S^n). Applying H_n gives a diagram of abelian groups and homomorphisms:



[2] We use the words *map* and *function* interchangeably.