SOLVING PROBLEMS USING ELEMENTARY MATHEMATICS

DAVID GAY

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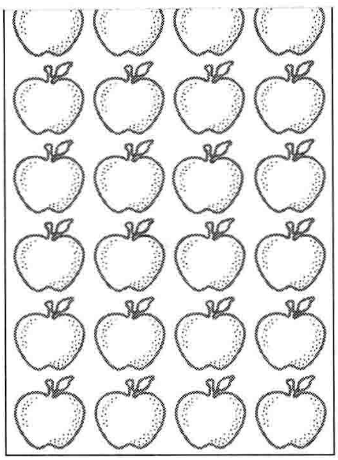

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To
Lonie, Adam,
and Katie

On the cover: "Nine" is an oil painting by San Francisco Bay Area artist Deborah Oropallo. The work is from a series inspired by Veronica O'Neill's book *Teaching Arithmetic to Deaf Children.* Deborah Oropallo's work was presented in the last Biennial Exhibition at New York's Whitney Museum. Her work may be seen in the collections of the Baltimore Museum of Art, Baltimore, Maryland; the La Jolla Museum of Contemporary Art, La Jolla, California; the Lannan Foundation, Los Angeles, California; and the Crocker Museum in Sacramento, California. Oropallo is represented by the Stephen Wirtz Gallery in San Francisco, California.

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FOREWORD

Mathematics is something that is meant to be done and enjoyed. It is a way to solve interesting problems that could not otherwise be solved, or to solve problems much more efficiently than they could otherwise be solved. It is not a spectator sport, nor is it a way to provide dreary, boring activities to keep children busy.

Unfortunately, much of school mathematics gives the impression that we teach it to keep children occupied so they won't have time to get into trouble: a sort of replacement for the nineteenth century practice of making samplers, except in the case of samplers the final product was beautiful and worth saving. A major goal of the National Council of Teachers of Mathematics *Standards for Teaching Mathematics,* and other similar reports, is to help students see that mathematics is exciting and useful by allowing them to actually do mathematics.

In this book David Gay makes the learner an active participant in doing mathematics. He helps the reader discover how mathematical thinking can be used to analyze and solve varied problems, some of which should be of interest to almost anybody, no matter what his or her previous background. He brings his extensive knowledge and creative imagination to elementary school mathematics and provides motivated and intellectually honest development of topics that should be taught to young children.

Prospective teachers who study this material and who do a reasonable number of the many problems and laboratory activities suggested in this book and its companion laboratory manual will derive a much better understanding of what mathematics is, why we teach it, and how they can teach it so children will learn and enjoy mathematics. Their pupils will think of mathematics as something to be done and enjoyed—as something they can figure out themselves (without memorizing formulas or procedures), and as something that is useful in solving problems that they wish to solve for their own benefit and for the benefit of others.

Stephen S. Willoughby

PREFACE

TO THE READER

A compelling reason why mathematics holds a central position in our educational system is that it can be useful in solving problems. Everybody, at one time or another, solves problems using mathematics. The more skill you have at solving problems with mathematics, the more options are open to you. The success of an engineer or scientist depends heavily on an ability to solve problems with mathematics. Although not everyone becomes an engineer or scientist, the recent proliferation of high-speed, electronic computers means that there are fewer jobs available to those without mathematical problem-solving skills.

"Skill" does not refer here to a tool to be used routinely or mechanically, without thinking. In the real world, routine problems such as "multiply these two numbers" or "simplify this algebraic expression" do not often appear. A real problem usually must be carried through a number of stages before it can be solved by activities such as "divide this fraction by that one." Problem-solving skills for the real world involve thinking and active involvement. These are skills of survival, and their use is very human.

The purpose of this book is to get you actively involved in solving problems with mathematics. This will be done in several ways. First, you will be introduced to some useful problem-solving strategies and given the opportunity to work with them. Secondly, you will encounter problems that are not routine, but realistic problems, which I hope you will perceive as really needing solutions. Thirdly, solutions to several of these problems will appear in the text, demonstrating to you (in an informal way) processes that lead to a solution, including the meanderings and dead-ends characteristic of any real problem-solving situation. Finally, you will see how mathematical ideas and techniques develop out of this process: a new idea is usually a consequence of solving a problem (or several problems), a new technique comes about because there is a need for it.

You intend to teach mathematics in an elementary or middle school; the mathematics of this text is closely related to the mathematics taught in such a school. However, you will have seen much of this before opening this book. How do you become involved when the problems are easy (to you) and the mathematical idea or technique is one that you already know? How do you think about a technique whose use has become routine? How do you think about an idea you may not have been encouraged to think about? When you become a teacher, you will want to be able to put yourself in your students' shoes; you will want to know what it is like to solve problems with limited knowledge. To turn the difficulty of thinking about something routine into an advantage, the text will ask you to solve a "simple" problem as if you were a person with a certain limited knowledge. For example, to introduce multiplication of several-digit whole numbers, a problem will appear that (you and I know) can be solved by the usual method. You will be asked to assume that you don't know the method and then to solve the problem using only other skills of arithmetic (addition, subtraction, single-digit multiplication, . . .).

Solving a problem using limited knowledge has positive consequences. First, you will learn directly that there are many ways to solve a problem, that there may be "unsophisticated" yet successful ways of solving a problem, and that deciding on the best way to solve a problem may depend on the person solving it. Secondly, it should make you aware of things you already know but haven't thought about lately. Thirdly, it should give you an understanding of how one piece of mathematics follows from another. This approach also makes sense to a future teacher: to nurture a mathematical idea in the mind of a fourth grader, it might be good if it first thrived in the mind of her teacher.

Here is what I want from your use of this book. I want you to begin to own, personally, the mathematical ideas that you once knew unthinkingly or only peripherally (and sometimes anxiously). I want you to begin to believe that mathematics is useful and use it. I want you to become competent and confident using mathematical ideas and techniques. I want you to be ready to learn how to get other persons actively involved in problem solving. I want you to have a blast solving problems.

Learning to solve problems can be frustrating, like learning to swim. Until you get that stroke down, you feel awkward and out of place. But when you grasp it, it's as if you and the water are one machine working together. Until you get the hang of solving problems, you can feel pretty inept. But when you finally succeed in solving a problem you haven't solved before, you will have a wonderful feeling of satisfaction. I hope that you will experience the joy of discovering mathematical facts you never knew before and of understanding for the first time how certain mathematical ideas and techniques fit together. As you solve more problems and gain more confidence in your mathematical abilities, you will have these good experiences more and more often.

Good luck and happy problem solving!

HOW EACH CHAPTER IS ORGANIZED

First Part of the Chapter Each chapter develops ideas and techniques that are related by a common theme. The first part of the chapter develops these in the following format: statement of a problem, solution of the problem, mathematical idea springing from the solution, exercises to try out the new idea. This format may repeat itself several times during a chapter. Each problem is chosen because it needs to be solved and because the techniques acquired up to that point for solving it are clumsy or inefficient. The solution to one problem will typically build on previous solutions and mathematical ideas.

The first part of the chapter is its core. Its style is informal, allowing ideas to germinate and evolve naturally.

Second Part of the Chapter The second part of the chapter embellishes the first. Its style is less leisurely and more condensed. It contains one or more sections of the following types. (Many chapters contain all types.)

- *Looking back* This section looks at the ideas of the first part of the chapter in a more rigorous or formal way than the first part does. It should provide you with a different perspective on the earlier material.
- *Looking ahead* This section relates the ideas of the first part to themes developed later in the text.
- *Extending ideas* This section carries some of the ideas of the first part further and puts them in a larger mathematical context. The ideas developed here will probably not be encountered later in the book.

- *Calculators and computers* This section discusses how calculators or computers (or both) can illustrate an idea, carry out a technique, or solve a problem. Chapter 1 has an introduction to calculators as well as to programming a computer in BASIC. (I encourage you to use calculators and computers to help you solve all the problems in this book, especially those that require a lot of computation.)

END-OF-CHAPTER FEATURES

- *Important ideas and techniques* This is a summary of the main concepts and techniques of the chapter. This should be useful for study and review.
- *Problem set* This is a collection of problems to solve to test your understanding and hone your problem-solving skills. The ideas and techniques of the chapter should be useful here. The problem set has two parts. Most of the problems in **Practicing Skills** are routine, needing only one or two steps to solve. A problem here may be like a problem solved in the chapter itself—in order to solve, you mimic the solution given in the text. The problems in the **Using Ideas** sections are more involved and may require you to use the ideas and techniques of the chapter in new ways. Or, you may need to develop a new, related idea or technique.
- *Three-chapter review* This is a set of problems for use as a "Sample Test" over the material of the previous three chapters. Solutions to the problems in these Reviews are in the back of the book.

OTHER FEATURES OF THE TEXT

STOP (Try this yourself) The stop sign occurs in the text after a statement of a problem, followed by an exhortation to try the problem before reading the text's solution. This is an important message! You need to get actively involved in solving problems in order to get the most out of this text.

Italics Key words appear in italics.

Boxes Boxed material highlights important definitions, ideas, and techniques. This should be useful for reference and review.

Boldface Color When a problem-solving strategy is used to solve a problem, its name will be printed in colored, boldface type. A list of all problem-solving strategies announced in this book is printed in the index under "Strategies for solving problems."

Exercises These occur throughout the text so that you can test your understanding immediately after an idea or technique has been presented. The answers to these exercises are in the back of the book.

Hands-on Activities Throughout the text you are encouraged to use objects to solve problems and enhance your understanding. Use these also to become acquainted with materials for teaching mathematics. Some of these items are readily available—stones, rulers, compasses, protractors. I may suggest that you trace others, cut them out, and tape them together.

Historical Comments These boxed items are meant to add human interest to the text's development and enlighten its ideas, without interrupting its flow.

STUDENT LAB MANUAL

The *Lab Manual* is a workbook-sized paperback containing laboratory activities to accompany specific topics in the text. The activities are designed especially to be used with hands-on materials.

The manual also contains things to cut out and assemble. Some of these can be used to accompany the text; others can be used to carry out the laboratory activities. The pages are perforated for easy removal.

Three of the lab activities involve the use of a computer spreadsheet.

STUDENT SOLUTIONS MANUAL

The *Solutions Manual* contains worked-out solutions to all exercises interspersed throughout the text and to many of the problems in the chapters' problem sets. This will provide you with more examples of problem solutions.

TO THE INSTRUCTOR

This Book and the NCTM Standards My selection of topics and My approach follow the recommendations made by the National Council of Teachers of Mathematics in its *Curriculum and Evaluation Standards for School Mathematics* (1989), commonly referred to as "The NCTM Standards." I feel that this text reinforces the goals of the Standards in the following particularly strong ways:

1. *Learning to value mathematics.* A strong attempt is made to connect the mathematics of this text with its uses in the real world. Real problems are presented that need to be solved. The mathematical ideas and techniques evolve from the need to create efficient solutions to these problems.
2. *Becoming confident in one's own ability.* To keep from overwhelming the reader with unnecessary terminology and symbolism, I try to introduce just those concepts and techniques that are needed and that emerge naturally from problem situations. I try to make the material appropriate for the reader's background and for how the reader will use it. The writing style is friendly and conversational; it is meant to help the reader become a participant in the development.
3. *Becoming a mathematical problem solver.* Problem solving is the heart of this book. The text integrates problem solving in its development: concepts emerge from problems and their solutions. I make every attempt to engage the reader in problem-solving activities and to provide all the aids I can for the reader to be successful in them.
4. *Learning to communicate mathematically.* Readers are encouraged to use certain strategies not only to solve problems but also to communicate ideas: draw a picture, make a model (use hands-on items), organize data in a table, make a graph, draw a histogram, and so on. These form part of a common language for users of this text. In part of each chapter's problem set (**Using Ideas**), the reader is asked to communicate his or her solution to each problem in the form of a written essay. In the *Lab Manual* are activities for several readers to work together solving problems and learning to communicate their ideas with each other.
5. *Learning to reason mathematically.* I make every effort to have the material make sense and hang together. Not only are there connections between the mathematics of the text and its real world uses, but also there are connections between mathematical ideas developed in the text itself. Making connections is an important aspect of reasoning. I provide arguments appropriate to the reader, plausible arguments that may not always be rigorous to a mathematician. I ask the readers to make similar connections and arguments in writing out the solutions to their problems.

Additional Themes Woven into the Text Several topics occur in several chapters as subsidiary themes.

- *Number line* This device is for visualizing the operations and the order relationships of numbers. Its use appears in chapters 3 through 11.
- *Algebra* This topic occurs in chapters 4 through 11 to show the connections between elementary mathematics and algebra.
- *Graphing* This theme is introduced in chapter 14 and is developed thereafter as a subtopic in the chapters on geometry, chapters 15 to 20.
- *Computational tools* The use of calculators and computers is introduced in chapter 1 and recurs throughout the text. Which tool is appropriate for which problem is frequently discussed.

The use of computers is integrated in the text in a variety of ways. Just enough BASIC commands are introduced in chapter 1 to solve some of the problems there. Additional BASIC commands are introduced as needed in later chapters. A section on *Logo* occurs in chapter 16, an early geometry chapter. The *Lab Manual* contains several activities using a computer spreadsheet. One of these can be used as an alternative to BASIC in chapter 12; this same activity may also be adapted for earlier use with the text. Another can be used with the sections on graphing. A third can be used with chapter 22, which deals with organization of data.

Possible Courses Using This Book You can use this text to design several different courses. There are two features of the book that can be particularly helpful to you in doing this.

First of all, the book contains material on a variety of topics, from chapters on whole number numeration and fractions to those on geometry and measurement, from chapters on estimation and graphing to those on probability and statistics.

Secondly, each chapter is designed so that you can choose the degree of informality or formality for treating the covered topics. The essential part of each chapter is the first part in which the main ideas are developed through solving many problems. In the second part there are many options for covering additional material. Some sections of this second part present the earlier material from a more formal or abstract viewpoint. Others present enrichment material. Still others present material that is developed through the themes mentioned above. Of course, you can select as many of these additional sections as you wish. If you want a more informal course, you may want to supplement the first part of the chapter with only one or two sections (or even none) of the second part. If you want a more formal course, you may want to supplement the first part with several sections of the second.

Here are some possible courses in which the first part of each chapter is covered plus an occasional section from the second part:

One semester (three hours per week) or one quarter (five hours per week) courses.

- Problem solving, whole numbers, fractions, and number theory: chapters 1–10.
- Problem solving, whole numbers, fractions, and probability: chapters 1–6, 8–10, 21.
- Problem solving, estimating, graphing, and geometry: chapters 1, 12–20.

Two semester (three hours each per week) or two quarter course (five hours each per week).

- First semester/quarter: problem solving, whole numbers, fractions, and integers: chapters 1–6, 8–11.
- Second semester/quarter: estimating, geometry, probability, and data presentation: chapters 12–13, 15–22.

Intense one semester course (four hours per week).

- Problem solving, whole numbers, fractions, and geometry: chapters 1–6, 8–10, 13, 15–20.

There are, of course, many other possibilities.

Instructor's Handbook This is a paperback that contains answers to all exercises and problems in the text that do not already appear at the end of the book.

ACKNOWLEDGMENTS This book is the result of twelve years of development. There were two stages. The first consisted of a lot of experimentation with materials for a mathematics course for prospective elementary school teachers at the University of Arizona. During this time, I can't remember having an inkling that a book might evolve from these materials. In the second stage, the text was prepared; it has been used in classes for four years and completely revised three times.

During both stages several persons made contributions to the final outcome of the project. My own students unwittingly encouraged me to experiment with new materials in the first place. I am particularly grateful to those who began their studies with me saying "I don't like mathematics and I was never any good at it" and left with "Hey! I *can* do it; math is neat!" Thanks also to the students who used the first versions of the text and put up with the misprints, omissions, and obscure explanations.

A second group to which I am indebted is the group of teaching assistants at the University of Arizona who worked with me in the experimental first stage or taught with the manuscript in the second stage. These persons were not only willing to try new ideas but also forced me to articulate mine better: Debi Anderson, Fernando Avila-Murillo, Teri Bennett, Jim Cain, Mirian Cuesta, Mark Dougan, Sil diGregorio, Steve Hammel, Gary Hudson, Steve Hughes, Grace Ikanaga, Jill Keller, Donna Krawczyk, Erich Kuball, Harry Miller, Burr Munsell, Diane Riggs, Steve Slonaker, Susan Taylor, Jon Thomsen, Steve Wheaton, and Mary Wheeler. I especially appreciate the assistance of Deborah Yoklic and James Abolt who are part of this group and who wrote new problems.

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It may not be traditional to thank one's competitors. However, I have found that several good books have helped me to find out what works for me and what are the important ideas. These are *From Sticks and Stones* by Paul Johnson, *Mathematics for Elementary Teachers* by Eugene Krause, and *Mathematics, an Informal Approach* by Albert Bennett and Leonard Nelson.

A book on solving problems needs good problems to solve. Rich sources of these I have used are *The Math Workshop: Algebra* by Deborah Hughes Hallett, *Using Algebra* by Ethan Bolker, *Make It Simpler* by Carol Meyer and Tom Sallee, *Sourcebook of Applications of School Mathematics* by Donald Bushaw et al., and *When Are We Ever Gonna Have to Use This?* by Hal Saunders. A book that was invaluable to me in the development of chapter 3 is *Number Words and Number Symbols* by Karl Menninger.

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David Gay

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