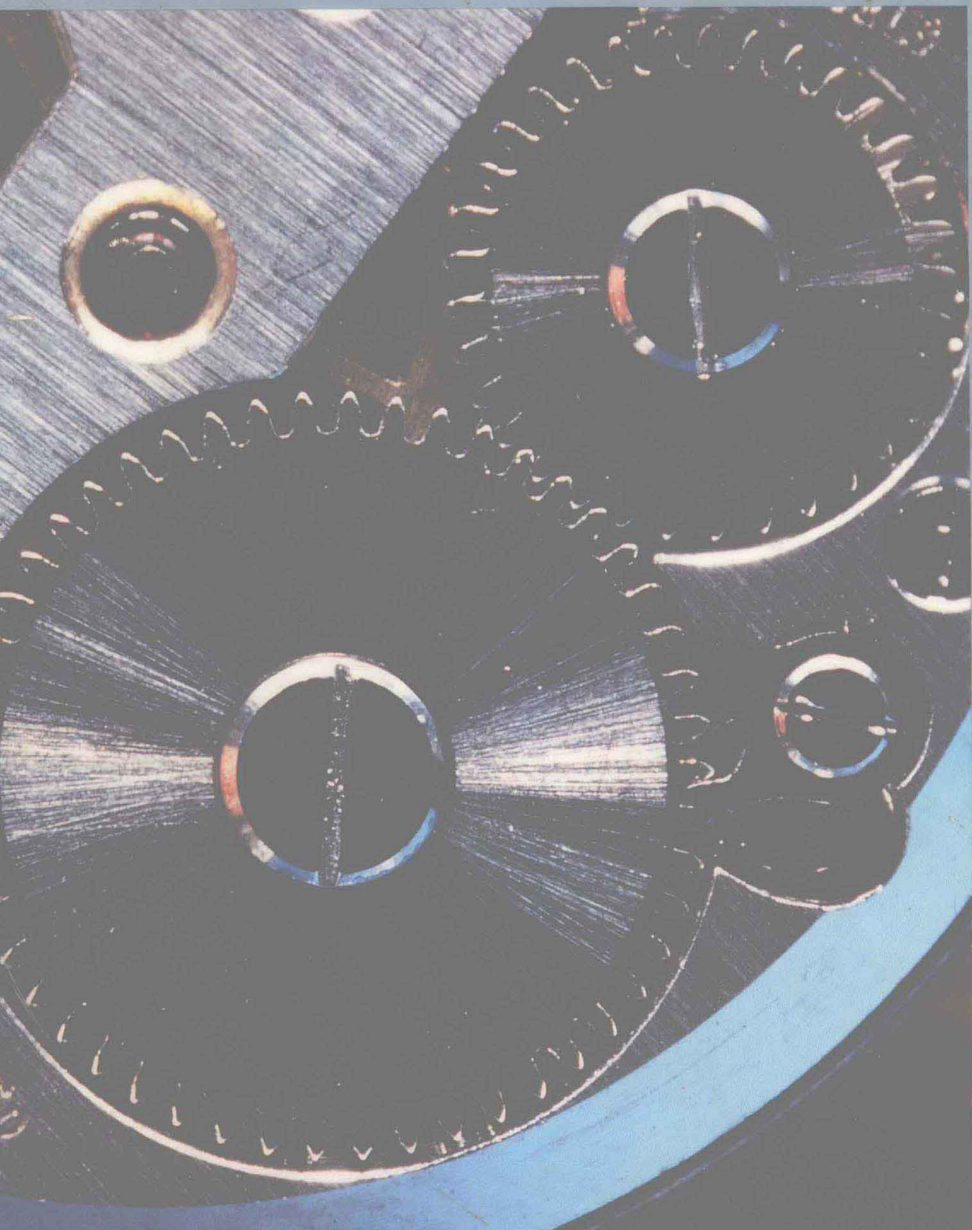
A FIRST COURSE in

# DIFFERENTIAL EQUATIONS



FIFTH EDITION

Dennis G. Zill

5th EDITION

# A FIRST COURSE IN DIFFERENTIAL EQUATIONS

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**PWS–KENT Publishing Company**
*Boston*

PWS-KENT
Publishing Company

20 Park Plaza
Boston, Massachusetts 02116

*Sponsoring Editor*: Steve Quigley
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*Developmental Editor*: Barbara Lovenvirth
*Manufacturing Coordinator*: Lisa Flanagan
*Cover and Interior Design*: Patricia Adams
*Cover Photo*: Gary Crallé/The Image Bank
*Interior Illustrations*: Deborah Schneck and Network Graphics
*Typesetting*: Polyglot Pte Ltd
*Cover and Insert Printing*: New England Book Components, Inc.
*Printing and Binding*: Arcata Graphics/Halliday



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*Insert 1*: p. i, top: Terje Rakke/The Image Bank; bottom: Andrea Pistolesi/The Image Bank; p. ii, top: Lawrence Hughes/The Image Bank; bottom: R. Hayo/The Image Bank; p. iii, top: NASA; bottom: The University Museum, University of Pennsylvania Neg. # T4-143c2; p. iv, top: Garry Gay/The Image Bank; bottom: Biophoto Associates, Science Source/Photo Researchers Inc.
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**Library of Congress Cataloging-in-Publication Data**

Zill, Dennis G.
A first course in differential equations / by Dennis G. Zill. — 5th ed.
p. cm.
Includes index.
ISBN 0-534-93154-5
1. Differential equations. I. Title.
QA372.Z54 1992
515′.35—dc20 92-16957
CIP

Printed in the United States of America
93 94 95 96 97 — 10 9 8 7 6 5 4 3 2

# PREFACE

My goal for the fifth edition was to achieve a balance between the concepts and presentation of materials that appealed to users of previous editions and the substantive changes made to strengthen and modernize the text. I feel I have achieved this balance, thus enabling the text to appeal to an even wider audience. Many of the additions and changes are the result of user and reviewer comments and suggestions. Moreover, these changes were made with the ultimate audience in mind—the students who will be using it. For this reason, solutions of every example have been read with an eye to improving their clarity. In various places I have either added further explanations where I thought they might be helpful, or "guidance boxes" at crucial points in the flow of the solution.

## NEW FEATURES

Some new features, that I hope students will find both interesting and motivational, have been added to the text. Essays written by mathematicians prominent in their specialty are included after Chapters 3, 4, 5, and 9. Each essay reflects the thoughts, creativity, and opinions of the individual author and is intended to enhance the material found in the preceding chapter. It is my hope that the addition of these essays will spark the interest of the students, encourage them to read mathematics, and help them to gain a realization that differential equations is not simply a dry collection of methods, facts, and formulas, but a vibrant field in which people can, and do, work.

Color inserts also have been added at intervals in the text. These pages consist of illustrations matched with photographs relating to some of the applications found in the text. I feel that these contribute to the visualization of the applications and thereby provide an added insight to students.

## CHANGES IN THIS EDITION

- Section 1.2 is now devoted solely to the concept of a differential equation as a mathematical model.
- The material on the differential equation of a family of curves has been deleted. A brief discussion of this concept is now given in Section 3.1 (Orthogonal Trajectories).
- The method of undetermined coefficients is one of the more controversial topics in a course in differential equations. In the last three

editions, this topic was developed from the viewpoint of using a differential annihilator as an aid in determining the correct form of a particular solution. While preparing this revision, a substantial number of reviewers indicated that the annihilator approach was too sophisticated for their students and requested a simpler rule-based approach. Other reviewers, however, desired no change. In order to satisfy each of these preferences, both approaches are presented in this edition. The instructor can now choose between undetermined coefficients based on the superposition principle for nonhomogeneous linear differential equations (Section 4.4) or those based on the concept of differential annihilators (Section 4.6). Moreover, the notion of a differential operator is now introduced in a separate section (Section 4.5) Thus, covering Section 4.4 does not preclude coverage of the otherwise useful concept of a differential operator.

- The review of power series in Section 6.2 has been greatly expanded. A discussion of the arithmetic of powers series (addition, multiplication, and division of series) has been added.
- A brief discussion of the "cover-up method" for determining coefficients in a partial fraction decomposition and a historical note on Oliver Heaviside have been added to Section 7.2.
- The discussion on the operational properties of the Laplace transform has now been divided into two sections: Section 7.3, Translation Theorems and Derivatives of Transforms, and Section 7.4, Transforms of Derivatives, Integrals, and Periodic Functions. This separation allows for a clearer, more comprehensive treatment of these topics.
- Gaussian elimination, in addition to Gauss-Jordan elimination, is now discussed in Section 8.4. The notation for indicating row operations on an augmented matrix has been improved.
- Chapter 9, "Numerical Methods for Ordinary Differential Equations," has been significantly expanded and partially rewritten. The Adams-Bashforth/Adams-Moulton multistep method has been added to Section 9.5. Section 9.6, Errors and Stability, and Section 9.8, Second-Order Boundary-Value Problems, are new to this edition.
- The BASIC programs previously in Chapter 9 have been deleted. Instead, we offer these programs, along with their FORTRAN and Pascal versions, on disk.
- Chapter 10, on partial differential equations and Fourier series, has been eliminated from this edition. It was the consensus of users that this material was unnecessary in a beginning course. The topics: Fourier series, partial differential equations, and solutions of boundary-value problems by separation of variables, integral transforms, and numerical methods, are covered in detail in the expanded version of this text, *Differential Equations with Boundary-Value Problems*, Third Edition.

- New problems, applications, illustrations, remarks, and historical footnotes, have been added throughout the text.

## SUPPLEMENTS AVAILABLE

### *For Instructors*

- *Complete Solutions Manual*, Warren S. Wright, Loyola Marymount University. This manual contains complete, worked-out solutions to every problem in the text.

### *For Students*

- *Student Solutions Manual*, Warren S. Wright. This manual provides solutions to every third problem in each exercise set.

### *Software*

- *Computer Programs*, C. J. Knickerbocker, St. Lawrence University. This disk contains a listing of computer programs for many of the numerical methods considered in this text. Each program is written in three languages: BASIC, FORTRAN, and Pascal. (For IBM or MAC.)
- *The Grapher*, Steve Scarborough, Loyola Marymount University. For the Macintosh, this flexible program can generate graphs of equations in rectangular and polar coordinates as well as curves defined parametrically. In addition, it has routines for graphing: series, interpolating polynomials, direction fields, and numerical solutions of differential equations. The updated version also contains numerical integration and numerical root finding.

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*Los Angeles*

# ACKNOWLEDGMENTS

It takes the teamwork of many people to compile a revision of a textbook. I am especially grateful to Barbara Lovenvirth, developmental editor, Patty Adams, production editor, Carol Reitz, copy editor, Warren and Carol Wright for their help in the manuscript preparation and for producing the excellent solutions manuals, John Ellison of Grove City College, Grove City, PA, for his valuable contribution to Chapter 9, and to the following reviewers for their counsel, comments, criticisms, and compliments:

William H. Beyer, *University of Akron*
Pasquale Condo, *University of Massachusetts, Lowell*
Philip S. Crooke, *Vanderbilt University*
John H. Ellison, *Grove City College*
Zdzislaw Jackiewicz, *Arizona State University*
Dr. Harry L. Johnson, *V.P.I. & S.U.*
Dr. C. J. Knickerbocker, *St. Lawrence University*
Carlon A. Krantz, *Kean College of New Jersey*
Stephen J. Merrill, *Marquette University*
Vivien Miller, *Mississippi State University*
Carol S. O'Dell, *Ohio Northern University*
Dr. Brian Rodrigues, *California State Polytechnic University*
Kimmo I. Rosenthal, *Union College*

I also want to recognize and extend my sincere appreciation to the following individuals for taking time out of their busy schedules to contribute the essays found in this edition:

Michael Olinick, Department of Mathematics and Computer Science, *Middlebury College*, ***Population Dynamics***

John H. Hubbard and Beverly West, Department of Mathematics, *Cornell University*, ***Chaos***

Gilbert N. Lewis, Department of Mathematical and Computer Sciences, *Michigan Technological University*, ***Tacoma Narrows Suspension Bridge Collapse***

C. J. Knickerbocker, Department of Mathematics, *St. Lawrence University*, ***Nerve Impulse Models***

D.G.Z.

# A FIRST COURSE IN DIFFERENTIAL EQUATIONS

# CONTENTS



[O] = *optional*

C H A P T E R

# INTRODUCTION TO DIFFERENTIAL EQUATIONS



## *Important Concepts*

**Ordinary differential equation**
**Partial differential equation**
**Order of an equation**
**Linear equation**
**Nonlinear equation**
**Solution**
**Trivial solution**
**Explicit and implicit solutions**
***n*-parameter family of solutions**
**Particular solution**
**Singular solution**
**General solution**
**Mathematical model**

The words *differential* and *equations* certainly suggest solving some kind of equation that contains derivatives. So it is; in fact, the preceding sentence tells the complete story about the course that you are about to begin. But before you start solving anything, you must learn some of the basic definitions and terminology of the subject. This is what Section 1.1 is all about. Section 1.2 is intended to be motivational. Why should you, an erstwhile scientist or engineer, study this subject? The answer is simple: Differential equations are the mathematical backbone of many areas of science and engineering. Hence, in Section 1.2 we examine, albeit briefly, how differential equations arise from attempts to formulate, or describe, certain physical systems in terms of mathematics.

## 1.1 BASIC DEFINITIONS AND TERMINOLOGY

In calculus you learned that given a function $y = f(x)$, the derivative

$$\frac{dy}{dx} = f'(x)$$

is itself a function of $x$ and is found by some appropriate rule. For example, if $y = e^{x^2}$, then

$$\frac{dy}{dx} = 2xe^{x^2} \quad \text{or} \quad \frac{dy}{dx} = 2xy. \tag{1}$$

The problem that we face in this course is not: given a function $y = f(x)$, find its derivative. Rather, our problem is: if we are given an equation such as $dy/dx = 2xy$, somehow to find a function $y = f(x)$ that satisfies the equation. In a word, we wish to *solve* differential equations.

**DEFINITION 1.1** Differential Equation

An equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (DE).

Differential equations are classified according to **type**, **order**, and **linearity**.

### Classification by Type

If an equation contains only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, it is then said to be an **ordinary differential equation** (ODE). For example,

$$\frac{dy}{dt} - 5y = 1$$

$$(y - x)\,dx + 4x\,dy = 0$$

$$\frac{du}{dx} - \frac{dv}{dx} = x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

are ordinary differential equations. An equation involving the partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation** (PDE). For example,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

are partial differential equations.

## Classification by Order

The order of the highest-order derivative in a differential equation is called the **order of the equation**. For example,

second-order first-order

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation. Since the differential equation $(y - x)\,dx + 4x\,dy = 0$ can be put into the form

$$4x\frac{dy}{dx} + y = x$$

by dividing by the differential $dx$, it is a first-order ordinary differential equation. The equation

$$a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

is a fourth-order partial differential equation.

Although partial differential equations are very important, their study demands a good foundation in the theory of ordinary differential equations. Consequently, in the discussion that follows we shall confine our attention to ordinary differential equations.

A general $n$th-order, ordinary differential equation is often represented by the symbolism

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0. \tag{2}$$

The following is a special case of (2).

## Classification as Linear or Nonlinear

A differential equation is said to be **linear** if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

It should be observed that linear differential equations are characterized by two properties:

**(i)** The dependent variable $y$ and all its derivatives are of the first degree; that is, the power of each term involving $y$ is 1.
**(ii)** Each coefficient depends on only the independent variable $x$.

An equation that is not linear is said to be **nonlinear**.

The equations

$$x\,dy + y\,dx = 0$$

$$y'' - 2y' + y = 0$$

and

$$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x$$

are linear first-, second-, and third-order ordinary differential equations, respectively. On the other hand,

coefficient depends on $y$ ⇩ power not 1 ⇩

$$yy'' - 2y' = x \quad \text{and} \quad \frac{d^3y}{dx^3} + y^2 = 0$$

are nonlinear second- and third-order ordinary differential equations, respectively.

## Solutions

As mentioned before, our goal in this course is to solve, or find solutions of, differential equations.

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**DEFINITION 1.2 Solution of a Differential Equation**
Any function $f$ defined on some interval $I$, which when substituted into a differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

---

In other words, a solution of an ordinary differential equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is a function $f$ that possesses at least $n$ derivatives and *satisfies* the equation; that is,

$$F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$$

for every $x$ in the interval $I$. The precise form of the interval $I$ is purposely left vague in Definition 1.2. Depending on the context of the discussion, $I$ could represent an open interval $(a, b)$, a closed interval $[a, b]$, an infinite interval $(0, \infty)$, and so on.

---

### EXAMPLE 1

Verify that $y = x^4/16$ is a solution of the nonlinear equation

$$\frac{dy}{dx} = xy^{1/2}$$

on the interval $(-\infty, \infty)$.

**Solution** One way of verifying that the given function is a solution is to write the differential equation as $dy/dx - xy^{1/2} = 0$ and then see, after substituting, whether the sum $dy/dx - xy^{1/2}$ is zero for every $x$ in the interval. Using

$$\frac{dy}{dx} = 4\frac{x^3}{16} = \frac{x^3}{4} \quad \text{and} \quad y^{1/2} = \left(\frac{x^4}{16}\right)^{1/2} = \frac{x^2}{4},$$

we see that

$$\frac{dy}{dx} - xy^{1/2} = \frac{x^3}{4} - x\left(\frac{x^4}{16}\right)^{1/2} = \frac{x^3}{4} - \frac{x^3}{4} = 0$$

for every real number. ■

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### EXAMPLE 2

The function $y = xe^x$ is a solution of the linear equation

$$y'' - 2y' + y = 0$$

on $(-\infty, \infty)$. To see this, we compute

$$y' = xe^x + e^x \quad \text{and} \quad y'' = xe^x + 2e^x.$$

Observe $\quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$

for every real number. ■

Notice that in Examples 1 and 2 the constant function $y = 0$, for $-\infty < x < \infty$, also satisfies the given differential equation. A solution of a differential equation that is identically zero on an interval $I$ is often referred to as a **trivial solution**.

Not every differential equation that we write necessarily has a solution.