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# Studies in Non-Linear Stability Theory

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# Preface

Non-linear stability problems formulated in terms of non-linear partial differential equations have only recently begun to attract attention and it will probably take some time before our understanding of those problems reaches some degree of maturity. The passage from the more classical linear analysis to a non-linear analysis increases the mathematical complexity of the stability theory to a point where it may become discouraging, while some of the more usual mathematical methods lose their applicability. Although considerable progress has been made in recent years, notably in the field of fluid mechanics, much still remains to be done before a more permanent outline of the subject can be established.

I have not tried to present in this monograph an account of what has been accomplished, since the rapidly changing features of the field make the periodical literature a more appropriate place for such a review. The aim of this book is to present one particular line of research, originally developed in a series of papers published in 'Journal de Mécanique' 1962—1963, in which I attempted to construct a mathematical theory for certain classes of non-linear stability problems, and to gain some understanding of the non-linear phenomena which are involved. The opportunity to collect the material in this volume has permitted a more coherent presentation, while various points of the analysis have been developed in greater detail. I hope that a more unified form of the theory has thus been achieved. The discussion of the existing literature has, in this context, been restricted to papers bearing directly on the subjects which are treated. However, a more extensive bibliography of research on non-linear stability problems has also been included.

The theory presented in this monograph is based essentially on two concepts: asymptotic expansions with respect to suitably defined small parameters and series expansions in terms of eigenfunctions. Although asymptotic approximations, in one form or another, have widely been used in the study of non-linear stability problems, the correct formulation and justification of the asymptotic method requires careful analysis, which I have attempted to provide. On the other hand, the concept of eigenfunction expansion seems not to have been recognized until now as a fundamental tool in the analysis of problems of the type considered here; it is this concept which appears to contribute most significantly to the clarification of the mathematical aspects of non-linear behaviour. Although the generalized eigenfunction expansions needed in the analysis still present certain difficulties, which the reader will find discussed as they arise, the concept seems a most promising element for future developments of the non-linear stability theory.

It is a pleasure to record my gratitude to Professor LEON TRILLING, who read the manuscript and suggested innumerable improvements of the text, and to Professor PAUL GERMAIN, whose continuous interest has stimulated the investigations reported here and who invited me to write this monograph.

Paris, April 1965 WIKTOR ECKHAUS

# Contents

Chapter 1

# INTRODUCTION

## 1.1 The notion of stability

The concept of stability is so familiar that any introduction may seem superfluous. Yet, on closer analysis, some discussion of the basic ideas appears necessary for the proper formulation of certain problems, in particular the non-linear ones.

The question of stability may, in simple terms, be stated as follows: Given an equilibrium state of a physical system, whose stability we wish to study, we consider a state near equilibrium and ask whether in the course of time the system will tend toward the given equilibrium state.

In somewhat more precise terminology, we shall call the departure from equilibrium a perturbation. The initial value of the perturbation is considered to be given in a problem of stability and the question is asked whether, starting from this initial value, the perturbation will tend to zero in the course of time. If this is the case the equilibrium state is called stable, it is called unstable otherwise.[1]

The mathematical formulation of the stability theory proceeds from the non-linear partial differential equations (or integro-differential equations) which describe the problem of mathematical physics under consideration, under the most general conditions. The unknown quantities are functions of three space coordinates and time, and are subject to some boundary conditions.

Certain special solutions of such a general problem are usually of particular interest; these are the solutions of permanent type (equilibrium states in previous terminology). The simplest among permanent-type solutions are the stationary solutions and we shall restrict our discussion to problems where such solutions exist. In certain cases we may, without worrying about unicity, chose for further study a very particular stationary solution, which depends upon only one or

[1] This definition of 'stability' and 'instability' will be employed throughout in the sequel. It should be noted however that terminology in the field of stability may vary from one author to another. The present definition agrees with LIN (1955) but disagrees for example with LA SALLE and LEFSCHETZ (1961).

two of the three space variables; occasionally even the trivial solution may be of interest. We shall call any of these stationary solutions a basic solution of the problem.

The stability problem for a basic solution is formulated by the following procedure: We suppose that at some initial instant the solution of the general problem consists of the basic solution plus a perturbation, while boundary conditions are not being perturbed. From the equations of the problem we obtain by substitution the equations governing the perturbation quantities. These are again partial non-linear differential equations (or integro-differential equations) for unknown quantities, which are functions of three space variables and time. The equations are homogeneous and are subject to homogeneous boundary conditions.

In the mathematical formulation of the stability problem the initial values of the perturbations are considered given, that is, they can be chosen at will. The question arises now, how should these initial values be chosen; that is, with respect to what perturbations should the stability be studied. To answer that question it becomes necessary to define more precisely the interpretation of the mathematical stability problem in terms of the physical mechanism under consideration.

In appearance the stability problem describes phenomena which may be realized in the physical world. However, this appearance is somewhat misleading and in an attempt to imagine an experiment that would correspond strictly to a problem of stability one faces some conceptiual difficulties. The basic solution (equilibrium state) whose stability is being studied, has no other a priori significance than that of a theoretical abstraction: it represents a possible functioning of the physical system in an idealized world. In studying the stability of the solution in question we admit in fact that conditions for its validity cannot entirely be realized in an experimental investigation. From this point of view the stability theory appears as a confrontation of the theoretician with the irregularities, imperfections and disturbances which are always present in the physical world.

Consider for example a problem of fluid mechanics concerned with the flow in a channel. The stationary laminar solution supposes perfectly smooth walls, perfect regularity of the machinery which sets the fluid in motion and absence of any sources of disturbances external to the apparatus. Needless to say, these conditions cannot be realized. We must therefore first analyse the stability of the solution with respect to perturbations whose nature and level of intensity may be considered representative for the fluctuations that are unavoidable in the experimental apparatus. If we find the solution to be stable, then the theory tells us that every perturbation, after being introduced, decays and tends to zero. In the actual apparatus perturbations are introduced not once, but continuously or intermittently throughout the time-history of the experiment. Thus the theoretical stability problem is of the nature of a stability test, from which we may conclude that the solution under consideration, although it will never be realized, does represent a good approximation for the functioning of the physical system. The amplitudes of the departures from the solution that will be observed in the experiment are determined by the level of fluctuations introduced in the apparatus and can thus presumably be reduced to an acceptable minimum.

Only in the case of stability with respect to the unavoidable level of perturbations can an experiment be performed corresponding to what is imagined in a stability problem: A perturbation of a given intensity and character can be created at a given moment and its evolution in time can be studied. In the case of instability with respect to the unavoidable level no experiment corresponding to the stability problem can be performed, since the initial departure from equilibrium can not be controled (exception being made for some extreemly simple systems, such as a pendulum held by hand in the neighbourhood of its unstable equilibrium). In fact, in the case of instability, the theory shows us that the solution under consideration has no chance of survival in the physical world and that the physical system will function according to some other solution. This *conclusion* is what can be verified in experiment.

Often the theory may predict that with respect to a given level of perturbations the solution is stable if a certain parameter, which can be varied at will, is within a certain range, and unstable otherwise. The boundary between the stable and unstable range is given by a critical value of the parameter and a confirmation of this critical value may be sought by an experimental investigation, which again must start in the stable range and consist of a slow variation of the parameter toward its critical value. In this, of course, care must be taken to maintain the perturbations at the level for which the theory has been established.

If the stability theory is indeed to account for the fluctuations which occur in the physical world, than it must be formulated in such way that the results can be interpreted from the point of view of some realizable experiment. This requirement means that perturbations must in principle be considered as entirely arbitrary (although suitably continuous) functions of the space coordinates. Moreover, it must be admitted that an arbitrary number of different perturbations can be introduced in an arbitrary order of succession an arbitrary number of times during the time-history of the phenomena. The study of a single perturbation as an initial value problem should be considered to be of the nature of a stability test of which the results must be interpreted with care. This is particularly pertinent in non-linear problems where, in certain cases of which we shall see later an example, successive introduction of two different perturbations may result in a different behaviour than what can be inferred from the separate study of the two perturbations.

On the other hand, we do have the liberty of fixing at will the level of perturbations (that is, their intensity defined in some suitable way) with respect to which we wish to study the stability. This reflects the assumption that by refining the experimental apparatus the 'unavoidable fluctuations' may be made smaller and smaller, while perturbations of any given intensity may be created by artificial means. In a limiting process we can even study perturbations of which the level is arbitrarily small, although it can never quite equal zero.

## 1.2 The nature of non-linear stability problems

The classical theory of stability is the linearized theory: we assume the perturbations arbitrarily small and in the equations and boundary conditions of the problem we neglect terms non-linear in the perturbation-quantities as compared to the linear ones. We then study the stability of the basic solution with respect to perturbations of infinitesimally low level. Obviously such perturbations are unavoidable in any experimental apparatus.

If now the physical system under consideration is 'autonomous', —that is, if the time variable does not appear explicitly in the equations and boundary conditions—, and if the basic solution is stationary, then the linearized equations of the stability problem admit solutions of which the behaviour with time is given by an exponential function. Instability is characterized by the existence of perturbations which grow without limit in the course of time; in the case of stability all perturbations tend to zero as time tends to infinity. A limiting case between stability and instability, which we shall call 'neutral stability', occurs when there exists a perturbation which, after being introduced, remains of essentially constant amplitude in the course of time, while all other possible perturbations tend to zero.

As an illustration let us consider the following simple problem which arises in connection with BURGERS' mathematical model of turbulence to be studied in some detail in chapter 5. Suppose that the function $\Phi(\eta, t)$ is a perturbation which in the linearized theory satisfies the equation

$$\frac{1}{R}\frac{\partial^2 \Phi}{\partial \eta^2} + \Phi - \frac{\partial \Phi}{\partial t} = 0; \quad 0 \leqq \eta \leqq 1 \tag{1.2.1}$$

and boundary conditions

$$\Phi(0, t) = \Phi(1, t) = 0 \tag{1.2.2}$$

$R$ is a parameter which may be varied at will.

The solutions of the problem are given by

$$\Phi_n = A_n e^{-\mu_n t} \sin(n + 1) \pi \eta; \quad n = 0, 1, 2, \ldots \tag{1.2.3}$$

where $A_n$ are arbitrary constants to be determined by the initial conditions, and where $\mu_n$ is given by

$$\mu_n = -\left[1 - \frac{\pi^2}{R} (n + 1)^2\right] \tag{1.2.4}$$

There exists thus a denumerable infinite sequence of possible perturbations $\Phi_n$. If $R < \pi^2$ all perturbations tend to zero and we have stability; if $R > \pi^2$ one or more perturbation increases without limits and we have instability. The limiting case of neutral stability occurs for $R = \pi^2$. In that case $\Phi_0$ remains constant in the course of time, while $\Phi_n$ for $n > 0$ tend to zero as $t \to \infty$.

We remark that in the linearized stability theory the superposition principle holds. Consequently, it makes no difference how many times and in what order of succession different perturbations have been introduced.

The linearized analysis may be considered as the first step in any stability theory; moreover, it is a natural starting point for the description and definition of non-linear problems. Obviously, non-linear stability problems arise whenever linearized theory is no longer adequate and non-linear terms of the equations have to be taken into account.

Let us consider first the case of stability in the sense of the linearized theory. We know that in this case infinitesimal disturbances decay to zero. Now infinitesimal disturbances are certainly unavoidable, but not all unavoidable disturbances may be considered infinitesimal. In other words, for the explanation of some physical phenomena we may be led to study the stability with respect to perturbations whose level can no longer be assumed infinitesimally small. The answer to the stability question depends then on the level of perturbation. We may expect that under certain conditions, while there is stability with respect to extremely small perturbations, an instability arises with respect to perturbations of level higher than a critical value. This behaviour, called 'instability to finite size perturbations', will in fact present itself frequently in the course of our study and may be considered one of the most typical phenomena associated with non-linear stability problems.

Consider next the case of instability with respect to infinitesimal perturbations. There is, in this case, an apparent contradiction within the linearized theory, since we assume very small perturbations and find that they grow without bounds. From the mathematical point of view the paradox may be circumvented by stating that we consider infinitesimal initial values and concluding, when we find that perturbations grow without limits, that the solution is only valid during some finite initial time interval. This naturally poses the question: what happens to the perturbations when they have grown so large that linearization can no longer be considered valid.

The behaviour of perturbations in the case of instability is a problem of fundamental interest and various types of behaviour are conceivable. We shall particularly be interested in the question of existence of non-linear bounds on the growth of perturbations. Thus, in studying the non-linear problem we may find that the amplitude of perturbation, instead of growing without limits, tends to a finite value as time tends to infinity. Moreover, this finite value may be small, although not infinitesimal. If that is the case, the basic solution under consideration, although unstable in the ordinary sense, still represents a good approximation of the functioning of the physical system. In constrast to the stable case, however, the level of departures from the solution observed in an experimental investigation will not be bound by the intensity of externally introduced fluctuations, but will be, so to speak, a property of the system itself.

The non-linear problems arising in cases of stability and of instability described above are in many respects complementary and we shall find that they may often be treated in one and the same analysis. The essential mathematical difficulty associated with these problems is illustrated by considering for example the case of instability.

If we assume that initial values of the perturbations are infinitesimally small, then the linearized theory may be considered an approximation valid during some finite initial time interval. This suggests that we may try to construct the solution of the non-linear problem by an iteration procedure, in which the linearized solution is taken as the first approximation. We would then obtain a series solution, every term of the series growing without limit as a function of time.[1] The convergence of the series can of course be investigated, although mathematical complexity may make such analysis prohibitive. Most certainly the series will not converge for all times and in particular for every fixed initial value, no matter how small, the series will not converge in the limit as time tends to infinity. On the other hand, it is this behaviour in the limit that we are interested in, since we want to find out whether the perturbation approaches a finite value. The nature of the problem

[1] For a recent example of this approach see for instance STRUMINSKY (1963a).

thus requires that, no matter what technique of approximation is used, certain essential non-linear effects be included in the first approximation.

The above discussion can be illustrated by a simple example: Consider the equation

$$\frac{dA}{dt} = A\left(1 - \frac{A^2}{\alpha^2}\right)\left(1 + \frac{A^2}{\beta^2}\right) \tag{1.2.5}$$

where $\alpha$ and $\beta$ are constants, $|\alpha| \ll |\beta|$, and where a real initial value $A(0) = a_0$ is given, such that $|a_0| \ll 1$. In a first approach we can, by an iteration procedure, construct a series solution in which the linearized solution is the first approximation and obtain a series of which every term grows without limit as a function of time. In fact, we have defined by this procedure an expansion valid in the vicinity of the singular point $A = 0$ of the equation. There is however another singular point, $A = \alpha$, which is of greater interest from the present point of view. What we wish is an approximation valid in a region containing both $A = 0$ and $A = \alpha$. We can obtain the desired result by an iteration procedure in which, as first approximation, we take the solution of the equation

$$\frac{dA}{dt} = A\left(1 - \frac{A^2}{\alpha^2}\right) \tag{1.2.6}$$

This leads to a series solution which can be shown to be convergent. The solution tends in the limit $t \to \infty$ to the singular point $A = \alpha$; inclusion of more terms in the series modifies the manner in which this point is approached.

In problems of greater complexity, when the unknown quantity is a function of space variables and time, the aim of the mathematical analysis is still essentially the same as in the simple problem above. We wish to construct a solution of which the region of validity includes vanishingly small perturbations as well as the equivalent of the singular point $A = \alpha$. However, in general we do not know a priori where this 'singular point' lies, nor are we even sure that it exists.

It should be clear now that non-linear problems of stability require special mathematical techniques. Before discussing these, however, it is useful to illustrate the nature of non-linear phenomena in stability from yet another point of view: that of experimental evidence. We shall briefly describe certain types of non-linear behaviour observed in experimental investigations in the field of fluid mechanics.

Perhaps the most classical problem of hydrodynamic stability is that of the flow between two coaxial rotating cylinders. First investigated theoretically and experimentally by G. I. TAYLOR (1923), it has been the subject of numerous studies ever since (see, for example, LIN (1955)). The predictions of the linearized theory have been confirmed by experiments. Moreover, in the case of instability it was found that under certain conditions a regular and time-independent pattern of secondary flow is established. This secondary flow can, no doubt, be interpreted as a perturbation, which in the linearized theory grows without limits, but which here, due to non-linear effects, has reached a final stationary amplitude.

A mathematically similar problem, with similar behaviour, arises in connection with the convective instability of a fluid layer heated from below. Experimental and theoretical investigations have been originated by BENARD (1901) and by RAYLEIGH (1916), recent developments of the linearized theory may be found in CHANDRASEKHAR (1961). Here again, one observes a reasonably regular pattern of secondary flow, which usually has the appearence of hexagonal cells and which may be interpreted as a manifestation of a non-linear limit to the growth of perturbations.

Cellular structure of more oscillatory character occurs in flame instability. Experimental evidence in this domain is provided by MARKSTEIN's (1951) investigations; theoretical studies have been orginated by LANDAU (see LANDAU and LIFSCHITS), further developments can be found in ECKHAUS (1961).

In all these cases, the phenomena ultimately observed in experiment still resemble the unstable perturbation predicted by linearized theory, only the amplitude is limited in size. It is for this reason that the behaviour may be interpreted in terms of a rather simple non-linear effect, that of a non-linear bound on the growth of perturbations. The situation changes radically when turbulence makes its appearence.

SCHUBAUER and SKRAMSTAD's (1947) famous experiments on boundary layer instability have revealed that unstable perturbations do appear as predicted by the linearized theory and grow for some time (or, more exactly, over some distance) roughly in accordance with the theory. Further development is however suddenly broken off by the appearance of a new phenomenon: the transition to turbulent motion. In other unstable flows turbulence may make its appearence in somewhat different and less explosive fashion (see for example SATO and KURIKI's (1961) experiments on wake-flow), but in all cases it is clear that the simple mechanism of a non-linear bound on the growth of linearly unstable perturbations is irrelevant to the essential phenomenon. On the contrary, the occurrence of turbulence is more in the nature of a new instability, of essentially non-linear character.

Admitting that turbulence occurs through some kind of instability, in certain cases of linearly stable flows we have examples of behaviour that we have called instability to finite size perturbations. Thus, in the experiments of SCHUBAUER and KLEBANOFF (1956) turbulence was triggered by excitations of sufficient intensity in a region of boundary-layer flow that is stable according to linearized theory. On the other hand, REYNOLDS' (1883) experiments, which lay at the very origin of the study of instability and turbulence, were concerned with flow in circular pipes. while linearized stability theory has not revealed an instability for fully developed pipe-flow. Yet turbulence has been observed under various circumstances and various explanations have been advanced (notably in terms of instability of the boundary layer in the inlet region). From the point of view of non-linear stability theory it should be noted that in modern investigations it was found possible to keep the flow laminar at higher and higher Reynolds-numbers by reducing the level of 'inevitable perturbations' to lower and lower values. In this connection we mention the experimental studies of LEITE (1959), whose conclusion was that '. . . transition to turbulent flow occurs whenever the amplitude of the disturbances exceeds a threshold value which increases with decreasing Reynolds-number . . .'

Theoretical research on non-linear stability problems in fluid mechanics has been pursued actively in recent years and has resulted in considerable progress. For an excellent account of what has been achieved the reader is referred to STUART's (1960a) review; more recent publications are included in the Bibliography. We shall have the opportunity to discuss briefly some of this work in chapter 9, in connection with the problem of Poiseuille flow.

## 1.3 Formal approach to stability theory

As we have seen in section 1.1, the theory of stability is concerned with homogeneous partial non-linear differential equations with homogeneous boundary conditions imposed on some limiting surfaces. In studying the non-linear problems in particular, our goal is to develop satisfactory methods of solution and analyse the behaviour of perturbations. But if we are concerned with methods rather then with quantitative results there is no need to confine the attention to a particular problem. On the contrary, we can study classes of mathematical problems without apparent links with physical systems, provided that these problems are chosen so as to be representative of physical stability problems of interest, and provided secondly that in the choice of perturbations we respect the considerations of section 1.1. This is the approach adopted in the present study. It was originally developed by this author in a series of papers (1962—1963) published in *Journal de Mécanique*, which have served as a basis for the chapters that follow.

The formal approach to stability theory has the advantage of greater generality and flexibility. Thus, within certain limits, we may chose the problems at will and follow a course of study which seems natural from the point of view of mathematical analysis: we start with simple problems and with the understanding and experience gained in their study proceed to more complicated ones.

It is with these considerations in mind that we shall first consider (chapters 2 to 5) a class of problems in one dimensional space. Those problems do not claim to reflect the functioning of any physical system, but as mathematical models they represent certain features of non-linear stability theory in its simplest form.

Our analysis will be based mainly on asymptotic developments with respect to suitably defined small parameters, and on expansions in series of eigenfunctions of the linearized theory. When we proceed to more realistic problems in two-dimensional space (chapters 6 to 8) we shall find the same mathematical tools useful, although in somewhat modified form. Indeed, the eigenfunction expansion in particular can be said to be characteristic of the method of analysis developed and utilised throughout this study.

In a final chapter we shall consider an application of the formal theory to the problem of stability of Poiseuille flow. This will allow us to confront our analysis with results obtained earlier by different methods.

A word should be said about the choice of the classes of problems studied below. We shall assume that boundary conditions are imposed on surfaces that are fixed in space. This leaves out of consideration problems associated with so-called 'free-boundaries', which, in this authors opinion, require a different approach. To simplify the equations we shall consider problems involving one unknown quantity and thus governed by a single non-linear partial differential equation. Furthermore, the equation to be considered will involve only a first order derivative with respect to time, which will appear linearly. Thus the problems to be studied are governed by non-linear 'diffusion-type' equations.

Meaningful stability problems associated with diffusion-type equations arise for instance in fluid mechanics, but it is not only the authors familiarity with this field which has led to that choice. In fact, the study of diffusion-type equations appears quite natural if one is interested in the stability of *stationary* solutions.

On the other hand, in most cases in which the governing equation contains higher order derivatives with respect to time, problems of stability and oscillations are interwoven and one is led to study at the outset the stability of *non-stationary* solutions of permanent type.

In the present study we shall eventually consider certain oscillatory permanent-type solutions and their stability, but these solutions will arise as by-products of the stability-analysis of stationary solutions, which will be our major concern.