Graduate Texts in Mathematics

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General Relativity for Mathematicians

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Preface

This is a book about physics, written for mathematicians. The readers we have in mind can be roughly described as those who:

1. are mathematics graduate students with some knowledge of global differential geometry
2. have had the equivalent of freshman physics, and find popular accounts of astrophysics and cosmology interesting
3. appreciate mathematical clarity, but are willing to accept physical motivations for the mathematics in place of mathematical ones
4. are willing to spend time and effort mastering certain technical details, such as those in Section 1.1.

Each book disappoints some readers. This one will disappoint:

1. physicists who want to use this book as a first course on differential geometry
2. mathematicians who think Lorentzian manifolds are wholly similar to Riemannian ones, or that, given a sufficiently good mathematical background, the essentials of a subject like cosmology can be learned without some hard work on boring details
3. those who believe vague philosophical arguments have more than historical and heuristic significance, that general relativity should somehow be "proved," or that axiomatization of this subject is useful
4. those who want an encyclopedic treatment (the books by Hawking–Ellis [1], Penrose [1], Weinberg [1], and Misner–Thorne–Wheeler [1] go further into the subject than we do; see also the survey article, Sachs-Wu [1]).
5. mathematicians who want to learn quantum physics or unified field theory (unfortunately, quantum physics texts all seem either to be for physicists, or merely concerned with formal mathematics).

While using this book in classes, we found that our canonical reader can learn nonquantum physics rather quickly. Indeed, equipped with geometric intuition and a facility with abstract arguments, he is in a position to deal directly with the general, currently accepted models used in relativity without being handicapped by the prejudices that inevitably come with years of Newtonian training in the standard physics curriculum. However, this shortcut does involve a price: one cannot really see the diversity of special cases behind the deceptively simple foundation without spending more time than a mathematics student normally can or should.

We have felt for a long time that a serious effort should be made by physicists to communicate with mathematicians somewhat along the line of this book. We started with the aim of keeping the physics honest, keeping the mathematics honest, and keeping the logical distinction between the two straight. But we were ill-prepared for the attendant trauma of such an undertaking. In particular, the third point proved to be a veritable nightmare. We managed to emerge from our many moments of doubt to complete this book with the original plan intact, not the least because we were sustained from time to time by the encouragement of some of our friends and colleagues, particularly S. S. Chern and B. O'Neill. Nevertheless, we are pessimistic about further attempts at explaining genuine physics to mathematicians using only prerequisites familiar to them.

Many people believe that current physics and mathematics are, on balance, contributing usefully to the survival of mankind in a state of dignity. We disagree. But should humans survive, gazing at stars on a clear night will remain one of the things that make existence nontrivial. We hope that at some point this book will remind you of the first time you looked up.

Through the several drafts of this book as classroom notes, we were fortunate to have the excellent secretarial assistance of Joy Kono, Nora Lee, and Marnie McElhiney. A philosophical remark from Professor S. S. Chern was responsible for an overhaul of our overall presentation. Many minor and quite a few major improvements were due to suggestions by J. Arms, J. Beem, K. Sklower and T. Langer. But for the warm hospitality of the DAMTP of Cambridge University and the unswerving support of Kuniko Weltin under rather trying circumstances, the final stage of the book-writing would have been interminable and insufferable. Finally, support from the National Science Foundation greatly facilitated the preparation of the manuscript. To all of them, we wish to express our deep appreciation.

Guidelines for the reader

1. All indented fine-print portions of the book are optional; they may be skipped without loss of mathematical continuity. Some of these fine-print paragraphs are proofs that we consider noninstructive. But the majority of them contain comments that presuppose a knowledge of physics (and on a few occasions, of mathematics) beyond the level of our formal prerequisites. Nevertheless, we urge the reader to at least glance through those dealing with physics; they may be read with the assurance that each has been revised many times to minimize distortion of the physics.
2. The remainder of the text should be treated as straight mathematics, though one should keep in mind the following peculiarity: There will be no all-encompassing mathematical abstractions; instead, the emphasis throughout is on simple definitions and propositions that have a multitude of physical implications.

   Physics attempts to describe certain aspects of nature mathematically. Now, nature is not a mathematical object, much less a theorem. There is no overriding mathematical structure that covers all of physics. Since the subject matter of this book is physics, the reader will find here not a coherent and profound mathematical study of general Lorentzian manifolds culminating in a *Hauptsatz*, but rather a disjointed collection of propositions about a special class of four-dimensional Lorentzian manifolds. Mathematics plays a subordinate role; it is a tool rather than the ultimate object of interest. For example, in Chapter 3 the emphasis is not on the coordinate-free version of Stokes' theorem, which is taken for granted. Instead, this theorem is used to define and analyze many physical concepts: the conservation of electric charge; the creation and annihilation of matter; the hypothesis that magnetic monopoles don't exist; relativistic versions of Gauss' law for electric flux, Faraday's law of magnetic induction, and Maxwell's displacement current hypothesis; the special relativistic laws for conservation of energy, momentum, and angular momentum; and so on. These concepts in turn apply to a very rich variety of known phenomena. Our aim in discussing the theorem is merely to indicate how it can manage to say so much about the world so concisely.

In brief: economy will be central, mathematical generality will be irrelevant.

The reader wishing to pursue the deeper mathematical theorems of relativity should consult Hawking–Ellis [1].

3. The expository style of the book is strictly mathematical: all concepts are explicitly defined and all assertions precisely proved. Now, in a serious physics text basic physical quantities are almost never explicitly defined. The reason is that the primary definitions are actually obtained by showing photographs, by pointing out of the window, or by manipulating laboratory equipment. The more mathematically explicit a definition, the less accurate it tends to be in this primary sense. The reader is therefore forewarned that on this one point we have intentionally distorted an essential feature of physics in order to accommodate the mathematician's intolerance of theorems about mathematically undefined terms.
4. The exercises at the end of each section are, at least in principle, an integral part of the text. We have been very conscientious in making sure that each is workable within a reasonable amount of time.
5. Chapters 0 through 5 are meant to be read consecutively. The remaining chapters are independent.

Contents

Preliminaries 0

This chapter is intended mainly to clear the boards for action. A reader with a solid background might try just skimming the chapter. Section 0.0 reviews some of the differential geometry we shall need. Section 0.1 gives some physics background. Section 0.2 gives an intuitive discussion of the transition from Newtonian physics to relativity. A reader who has never studied relativity should work all the exercises for Section 0.2.

0.0 Review and notation

This section sets the notation. Definitions not explicitly stated and theorems not explicitly proved are all discussed, for example, in the text by Bishop and Goldberg [1], referred to as Bishop–Goldberg throughout. We follow the Bishop–Goldberg notation as closely as feasible.

0.0.1 Sets, maps, and topology

Suppose A and B are sets, and $i\colon A \to B$ is a map, the image of $a \in A$ is written either ia or $i(a)$. For example, suppose C is a set and $k\colon B \to C$ is a map; then $(k \circ i)(a) = (k \circ i)a = k(ia) = kia$ with $k(ia)$ preferred. Suppose D is a subset of A. We write $D \subset A$ and write $A - D = \{a \in A \mid a \notin D\}$; $i|_D$ is the restriction of i to D. Suppose $E \subset B$; we write $i^{-1}E \subset A$ for the complete inverse image. Suppose A above is a topological space; then D^- and ∂D will denote the closure and boundary, respectively, of D.

$\mathbb{Z}$ denotes the integers and $\mathbb{R}$ the reals. If $\mathscr{E} \subset \mathbb{R}$ is connected and open, we sometimes write $\mathscr{E} = (a, b)$, with $a = -\infty$ and/or $b = \infty$ allowed. $f\colon \mathscr{E} \to \mathbb{R}$ is called *positive affine* iff $fu = cu + d$, where $c > 0$ and $d \in \mathbb{R}$.

0.0.2 Tensor algebra

If V is a vector space (always understood to be over $\mathbb{R}$), V^* denotes its dual. The image of $v \in V$ under $\omega \in V^*$ is denoted by $\omega(v)$ or ωv. By a *subspace* of V we mean a vector subspace. If $V_1, \ldots, V_N$ are finite dimensional vector spaces, then $V_1 \oplus \cdots \oplus V_N$ will denote their direct sum and $V_1^* \otimes \cdots \otimes V_N^*$ the vector space of multilinear maps $V_1 \times \cdots \times V_N \to \mathbb{R}$. The space of (r, s) *tensors* over V_1 is $T_s^r(V_1) = V_1 \otimes \cdots \otimes V_1 \otimes V_1^* \otimes \cdots \otimes V_1^*$, where there are r unstarred and s starred factors. (r, s) is the *type* of each tensor in $T_s^r(V_1)$. Suppose $S \in T_s^r(V_1)$ and $T \in T_q^p(V_1)$; then $T \otimes S \in T_{s+q}^{r+p}(V_1)$ denotes the tensor product.

We are following the convention of Bishop–Goldberg in placing the contravariant variables in front of all the covariant variables for each tensor in $T_s^r(V_1)$. This is aimed at facilitating any discussion concerning tensors when no mention of indices is allowed.

0.0.3 Inner products

Let V be a finite dimensional vector space. A nondegenerate symmetric bilinear form g on V is called an *inner product* on V (Bishop–Goldberg 2.21). Let $S = \{W \mid W \text{ is a subspace of } V \text{ and } g|_W \text{ is negative definite}\}$. The *index I* of g is the integer $I = \max_{W \in S}$ (dimension W). Define the *norm* of $v \in V$ as $|v| = [|g(v, v)|]^{1/2}$; define $v \in V$ as a *unit* vector iff $|v| = 1$; define $v, w \in V$ as *orthogonal* iff $g(v, w) = 0$.

Let $N = \dim V$, $B = (e_1, \ldots, e_N)$ be an ordered basis of V, and $(\varepsilon^1, \ldots, \varepsilon^N)$ be the dual basis of V^*. B is called ("ordered," "semi-") *orthonormal* iff $g = \sum_{A=1}^{N-I} \varepsilon^A \otimes \varepsilon^A - \sum_{A=N-I+1}^{N} \varepsilon^A \otimes \varepsilon^A$, where the appropriate sum is zero if $I = 0$ or $I = N$. Equivalently, B is orthonormal iff: $g(e_A, e_A) = 1$ for $1 \leq A \leq N - I$, $g(e_A, e_A) = -1$ for $N - I + 1 \leq A \leq N$, and $g(e_A, e_B) = 0$ for $A \neq B$. A basis of pairwise orthogonal unit vectors can always be made an orthonormal basis by appropriate reordering. If $e \in V$ is a unit vector, there exists an orthonormal basis that contains e.

We shall call the pair (V, g) a *Lorentzian vector space* and g a *Lorentzian inner product* iff $\dim V \geq 2$ and $I = 1$.

This is the case of main interest. The reader should not assume it is essentially similar to the positive definite case. The differences are central in physics, as the rest of this book shows. For example, suppose g is an inner product on V. The subset $\{v \in V \mid g(v, v) < 0\}$ has two connected components iff (V, g) is Lorentzian. Locally, these components correspond to the physical past and physical future. When the algebraic structure of a Lorentzian (V, g) is unwrapped from tangent spaces into a manifold, a rich structure results (Penrose [1], Hawking and Ellis [1]). See Optional exercises 8.3.

0.0.4 C^∞ *Manifolds and maps*

Unless specifically denied, all manifolds, all objects on them, and all maps from one manifold into another will be C^∞; however, we sometimes redundantly write "a C^∞ manifold," and so on, for emphasis. A manifold M introduced by a definition need not be connected, but will always be finite-dimensional, real, Hausdorff, and paracompact. Throughout the remainder of this book, M is a manifold. M_x denotes the tangent space at $x \in M$. The *tangent bundle* TM is $\{(x, X) \mid x \in M \text{ and } X \in M_x\}$ with its standard C^∞ manifold structure (Bishop–Goldberg 3A); the projection $\Pi: TM \to M$ has the rule $\Pi(x, X) = x$. As in Bishop–Goldberg, M_x will be identified with the *fibre* $\Pi^{-1}x$ over x.

Let N be a manifold and $\phi: N \to M$ be a map. Then the map $\phi_*: TN \to TM$ between tangent bundles denotes the differential and ϕ^* denotes the pullback. Thus $(\phi \circ \psi)^* = \psi^* \circ \phi^*$. ϕ is an *immersion* iff $\forall n \in N$, ϕ_* restricted to N_n is one-one. An immersion ϕ is an *imbedding* iff ϕN, with the topology induced by that of M, is homeomorphic to N under ϕ. Then ϕ is one-one and ϕN is called an *imbedded submanifold.* Any open subset of M is an imbedded submanifold. A *diffeomorphism* is an onto imbedding.

0.0.5 *Tensor fields*

Let $T_s{}^rM$ be the bundle of (r, s) tensors over M, and $P: T_s{}^rM \to M$ be the standard projection (Bishop–Goldberg 3A). An (r, s)*-tensor field* $\boldsymbol{B}$ on $\mathscr{U} \subset M$ is a map $\boldsymbol{B}: \mathscr{U} \to T_s{}^rM$ such that $P \circ \boldsymbol{B} =$ identity on $\mathscr{U}$. Thus for each $x \in \mathscr{U}$, $\boldsymbol{B}x \in T_s{}^r(M_x)$. If $\mathscr{U}$ is a submanifold of M, then $\boldsymbol{B}$ being C^∞ makes sense, and this property will then be automatically assumed by our convention.

We follow the standard definitions of the usual tensor formalism (Bishop–Goldberg 2, 3, 4). For example, suppose $f: M \to \mathbb{R}$ is a function on M; $\boldsymbol{V}$ and $\boldsymbol{W}$ are vector fields on M; and $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are 1-forms on M. Then: (a) $L_{\boldsymbol{V}}$ denotes the Lie derivative with respect to $\boldsymbol{V}$; (b) $\boldsymbol{V}f = df(\boldsymbol{V}) = L_{\boldsymbol{V}}f$ is a function on M; (c) $[\boldsymbol{V}, \boldsymbol{W}]$ denotes the Lie bracket so that $[\boldsymbol{V}, \boldsymbol{W}] = L_{\boldsymbol{V}}\boldsymbol{W}$; (d) $\boldsymbol{\varphi} \wedge \boldsymbol{\psi} = \frac{1}{2}(\boldsymbol{\varphi} \otimes \boldsymbol{\psi} - \boldsymbol{\psi} \otimes \boldsymbol{\varphi})$; and (e) $2d\boldsymbol{\varphi}(\boldsymbol{V}, \boldsymbol{W}) = \boldsymbol{V}\boldsymbol{\varphi}(\boldsymbol{W}) - \boldsymbol{W}\boldsymbol{\varphi}(\boldsymbol{V}) - \boldsymbol{\varphi}([\boldsymbol{V}, \boldsymbol{W}])$. A q-form $\boldsymbol{\tau}$ on M is called *closed* iff $d\boldsymbol{\tau} = 0$, *exact* iff there is a $(q-1)$-form $\boldsymbol{\mu}$ on M such that $\boldsymbol{\tau} = d\boldsymbol{\mu}$. An exact q-form is closed.

We use the usual swindle for domains of definition. For example, let $\boldsymbol{g}$ be a (0, 2) tensor field on M, and $\boldsymbol{V}$ be a vector field on M; suppose $W \in M_x$ for some $x \in M$. Then $\boldsymbol{g}(\boldsymbol{V}, W)$ means $\boldsymbol{g}x(\boldsymbol{V}x, W) \in \mathbb{R}$ and $\boldsymbol{g}(\cdot, W)$ means $\boldsymbol{g}x(\cdot, W) \in M_x^*$. As another example, if $\boldsymbol{U}$ is a vector field defined on an open submanifold $\mathscr{N}$ of M, then $\boldsymbol{g}(\boldsymbol{U}, \boldsymbol{V})$ means $\boldsymbol{g}|_{\mathscr{N}}(\boldsymbol{U}, \boldsymbol{V}|_{\mathscr{N}})$, which is a function on $\mathscr{N}$.

An n-dimensional manifold M is called *orientable* iff there is a nowhere zero n-form $\boldsymbol{\omega}$ on M; any such $\boldsymbol{\omega}$ is called a *volume element* and determines an *orientation* (Bishop–Goldberg 3C and p. 185). If M is an oriented manifold and $\mathscr{U} \subset M$ is open, we always assign the consistent orientation to $\mathscr{U}$.

If, furthermore, $\partial\mathscr{U}$ is a submanifold of M, then $\partial\mathscr{U}$ inherits an *induced orientation* from M in the following manner: if $x \in \partial\mathscr{U}$ and $\{x^1, \ldots, x^n\}$ are coordinate functions in an open set $\mathscr{A}$ containing x such that $\mathscr{U} \cap \mathscr{A} = \{x^1 < 0\}$ and $\boldsymbol{dx}^1 \wedge \cdots \wedge \boldsymbol{dx}^n$ is consistent with the orientation of M, then $\boldsymbol{dx}^2 \wedge \cdots \wedge \boldsymbol{dx}^n$ restricted to $\partial\mathscr{U}$ is consistent with the induced orientation on $\partial\mathscr{U}$.

0.0.6 Curves

Let $\mathscr{E} \subset \mathbb{R}$ be an interval, which may be infinite, and $\gamma\colon \mathscr{E} \to M$ a map. γ will always be understood to be C^∞ in the following sense: there exists an open set $\hat{\mathscr{E}} \subset \mathbb{R}$ containing $\mathscr{E}$ and a C^∞ map $\hat{\gamma}\colon \hat{\mathscr{E}} \to M$ such that $\hat{\gamma}|_{\mathscr{E}} = \gamma$. Such a C^∞ map $\gamma\colon \mathscr{E} \to M$ is called a *curve* in M. We denote the inclusion function $\mathscr{E} \to \mathbb{R}$ by s, t, or u and the distinguished vector field on $\mathscr{E}$ by d/ds, and so on. For example, $\boldsymbol{du}(d/du) = 1$. For each $u \in \mathscr{E}$, $\gamma_* u$ denotes the tangent vector at γu; thus $\gamma_* u = [\gamma_*(d/du)](u) \in M_{\gamma u}$.

A curve $\gamma\colon \mathscr{E} \to M$ is called *inextendible* iff any other curve $\zeta\colon \mathscr{F} \to M$ satisfying $\mathscr{E} \subset \mathscr{F}$ and $\zeta|_{\mathscr{E}} = \gamma$ is the curve $\gamma\colon \mathscr{E} \to M$ itself. A curve $\zeta\colon \mathscr{F} \to M$ is called an *(orientation-preserving) reparametrization* of $\gamma\colon \mathscr{E} \to M$ if there exists an onto map $\alpha\colon \mathscr{E} \to \mathscr{F}$ with positive derivative such that $\gamma = \zeta \circ \alpha$. If α is positive affine, then ζ is called a *positive affine reparametrization* of γ.

If X is a vector field on M, the maximal integral curve of X through $x \in M$ is the unique curve $\gamma\colon (a, b) \to M$, $-\infty \leq a < b \leq \infty$, such that (a) $\gamma 0 = x$; (b) $\gamma_* u = X(\gamma u)\ \forall u \in (a, b)$; and (c) γ is inextendible (Bishop–Goldberg 3.4). The flow of X will be denoted by $\{\mu_s\}$. For example, if X is complete, $\mu_s\colon M \to M$ is obtained by moving each $x \in M$ s parameter units along the maximal integral curve through x (Bishop–Goldberg 3.5).

0.0.7 Metrics and isometries

Let $\boldsymbol{g}$ be a symmetric (0, 2) tensor field on M. $\boldsymbol{g}$ is called a *metric tensor with index I on M* iff $\boldsymbol{g}x$ is nondegenerate and index $(\boldsymbol{g}x) = I\ \forall\, x \in M$. Then $(M, \boldsymbol{g})$ is called a *Riemannian* manifold iff $I = 0$, *semi-Riemannian* otherwise. We will call a semi-Riemannian manifold *Lorentzian* iff $I = 1$ and the dimension of M is at least 2.

Let $(M, \boldsymbol{g})$ and $(N, \boldsymbol{h})$ be Riemannian or semi-Riemannian manifolds. A map $\phi\colon M \to N$ is called an *isometry* iff ϕ is one-one, onto, and $\phi^*\boldsymbol{h} = \boldsymbol{g}$. Then ϕ is a diffeomorphism. $(M, \boldsymbol{g})$ is then called *isometric* to $(N, \boldsymbol{h})$ under ϕ. A map $\psi\colon M \to N$ is defined as a *local isometry* iff $\psi^*\boldsymbol{h} = \boldsymbol{g}$.

0.0.8 Geodesics

Throughout Section 0.0.8, $(M, \boldsymbol{g})$ is a Riemannian or semi-Riemannian manifold. The *Levi–Civita connection D* of $(M, \boldsymbol{g})$ is that ("linear," "affine") connection on M characterized by: (a) *symmetry*, $D_V W - D_W V = [V, W]$ for all vector fields V, W on M; and (b) *compatibility*, $D_V \boldsymbol{g} = 0$ for all such V (Bishop–Goldberg 5.11). A curve $\gamma\colon \mathscr{E} \to M$ is a *geodesic of* $(M, \boldsymbol{g})$ iff it is a geodesic of D on M (Bishop–Goldberg 5.12). We shall not count a constant curve, which has $\gamma\mathscr{E} = x \in M$, as a geodesic. If γ is a geodesic of (M, g), there is an $a \in \mathbb{R}$ such that $g(\gamma_* u, \gamma_* u) = a \ \forall u \in \mathscr{E}$. γ is called a *maximal* (or *inextendible*) *geodesic* iff it is both a geodesic and an inextendible curve. Note that if a geodesic γ is a reparametrization of another geodesic ξ, then γ is necessarily a positive affine reparametrization of ξ. Let X be a nowhere zero vector field. X is called a *geodesic vector field* iff $D_X X \equiv 0$. Thus X is geodesic iff each of its integral curves is a geodesic.

The *exponential map* $\exp_x$ at $x \in M$ maps a subset $\mathscr{U}_x \subset M_x$ into M as follows. The zero vector $0 \in M_x$ is in $\mathscr{U}_x$ and $\exp_x 0 = x$. A nonzero vector $X \in M_x$ is in $\mathscr{U}_x$ iff there is a geodesic γ: $[0, 1] \to M$ such that $\gamma 0 = x$ and $\gamma_* 0 = X$. For $X \in \mathscr{U}_x$, $X \neq 0$, γ is unique and $\exp_x X = \gamma 1$. $\mathscr{U}_x$ is open and $\exp_x$ is C^∞. For each $x \in M$, there is an open neighborhood $\mathscr{V}_x \subset \mathscr{U}_x$ of 0 such that $\exp_x|_{\mathscr{V}_x}$ is a diffeomorphism. (M, g) is *complete* iff $\mathscr{U}_x = M_x \forall x \in M$. (M, g) is complete iff every geodesic γ: $\mathscr{E} \to M$ can be extended to a geodesic $\mathbb{R} \to M$ (Bishop–Goldberg 5.13).

0.0.9 Bases and coordinate maps

Assume dimension $M = n \geq 1$. An ordered set $\{X_1, \ldots, X_n\}$ of vector fields on M is called a basis of vector fields on M iff $\{X_A x\}$ is a basis of $M_x \forall x \in M$. A basis $\{\omega^A\}$ of 1-forms on M is defined similarly. Bases $\{X_A\}$ and $\{\omega^A\}$ are called *dual* iff $\omega^B X_A = \delta_A{}^B \ \forall A, B \in \{1, \ldots, n\}$. Any basis uniquely determines a dual basis. If M is oriented, we assign the consistent orientation to each tangent space; unless explicitly denied, each basis used will then have the consistent orientation. A basis $\{X_A\}$ on a Riemannian or semi-Riemannian manifold (M, g), and its dual, are called *orthonormal* iff $\{X_A x\}$ is an orthonormal basis of $M_x \forall x \in M$ (cf. Exercise 0.0.15). On a given M there usually does not exist a basis of vector fields or 1-forms. However, one can always find such a basis in each coordinate neighborhood, and if g is also given, one can even choose this basis to be orthonormal.

We define $\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R}$, where there are N factors. u^A: $\mathbb{R}^N \to \mathbb{R}$ denotes projection onto the Ath factor. Thus $\{du^A\}$ is a basis of 1-forms on any open submanifold of $\mathbb{R}^N$; the dual basis will be denoted by $\{\partial_A\}$. If $\mathscr{U} \subset M$ and x: $\mathscr{U} \to \mathbb{R}^N$ is a coordinate map, $x^A = u^A|_{x\mathscr{U}} \circ x$ denotes the *Ath coordinate function*. The basis on $\mathscr{U}$ dual to $\{dx^A\}$ will be denoted also by $\{\partial_A\}$.

The *unit* $(N - 1)$*-sphere* $(\mathscr{S}^{N-1}, h, \zeta)$ is $\mathscr{S}^{N-1} = \{x \in \mathbb{R}^N | \ |x| = 1\}$, regarded as a C^∞ manifold, together with the standard induced metric h on $\mathscr{S}^{N-1}$ and the standard volume element ζ on $\mathscr{S}^{N-1}$. Thus if I: $\mathscr{S}^{N-1} \to \mathbb{R}^N$ is the inclusion, $h = I^*(\sum_{A=1}^N du^A \otimes du^A)$. Note that $\mathscr{S}^0$ is just the two points $\{-1, 1\} \subset \mathbb{R}$.

EXERCISE 0.0.10

Let V be a finite dimensional vector space. When V is regarded as a C^∞ manifold, it can be canonically identified with any of its tangent spaces. A basis-free method is part (a) following. (a) Regard $\omega \in V^*$ as a function $\tilde{\omega}$: $V \to \mathbb{R}$. Show that for each $v \in V$ there is precisely one isomorphism $\phi_v \colon V_v \to V$ such that $\omega(\phi_v w) = d\tilde{\omega}(w) \forall w \in V_v$ and $\omega \in V^*$. (b) Let g be an inner product on V, and $\tilde{g}\colon V \to \mathbb{R}$ be the function determined by $\tilde{g}(v) = g(v, v)$. Show $d\tilde{g}(w) = 2g(\phi_v w, v) \forall w \in V_v$ and $v \in V$. (c) Let (V, g) be a Lorentzian vector space and define a (0, 2) tensor field $\mathbf{g}$ on V by $\mathbf{g}(w, z) = g(\phi_v w, \phi_v z) \forall v \in V$ and $w, z \in V_v$. Show $(V, \mathbf{g})$ is a Lorentzian manifold.

EXERCISE 0.0.11

Let V be an N-dimensional vector space, g be an inner product on V, and $W \subset V$ be a K-dimensional subspace. We define $W^\perp = \{v \in V \mid g(v, w) = 0 \, \forall w \in W\}$; if w spans W, we shall also write $w^\perp \equiv W^\perp$. Show: (a) $W^\perp$ is an $(N - K)$-dimensional subspace. (b) $W^{\perp\perp} = W$. (c) $V = W \oplus W^\perp$ iff $g|_W$ is nondegenerate.

EXERCISE 0.0.12

If M is a manifold, show: (a) $\mathscr{U} \subset TM$ open implies $\Pi\mathscr{U} \subset M$ is open; (b) $\mathscr{V} \subset M$ open implies $\Pi^{-1}\mathscr{V} \subset TM$ is open.

EXERCISE 0.0.13

(a) Show that for Riemannian or semi-Riemannian manifolds, the relation "is isometric to" is an equivalence relation. (b) Let (M, g) and $(\hat{M}, \hat{g})$ be Riemannian or semi-Riemannian manifolds. Show that $\phi\colon M \to M$ is a local isometry iff each $x \in M$ has an open neighborhood $\mathscr{U} \subset M$ such that $(\mathscr{U}, g|_{\mathscr{U}})$ is isometric to $(\phi\mathscr{U}, \hat{g}|_{\phi\mathscr{U}})$ under $\phi|_{\mathscr{U}}$. (c) Let $\phi\colon M \to \hat{M}$ be a local isometry as in (b) and let $\gamma\colon \mathscr{E} \to M$ be a geodesic of (M, g). Show that $\hat{\gamma} = \phi \circ \gamma$ is a geodesic of $(\hat{M}, \hat{g})$. (d) Show that the set $\mathscr{G}M$ of isometries of a Riemannian or semi-Riemannian manifold (M, g) onto itself forms a group.

EXERCISE 0.0.14

Let V be a finite dimensional vector space and $\phi\colon V \to V^*$ a given isomorphism. (a) Show that for $r, s \in \mathbb{Z}$, $r > 0$, $s \geq 0$, ϕ can be extended uniquely to an isomorphism (to be denoted by the same symbol) $\phi\colon T_s{}^r(V) \to T_{s+1}^{r-1}(V)$ such that $\phi(v_1 \otimes \cdots \otimes v_r \otimes \omega^1 \otimes \cdots \otimes \omega^s) = v_1 \otimes \cdots \otimes v_{r-1} \otimes \phi(v_r) \otimes \omega^1 \otimes \cdots \otimes \omega^s$, $\forall v_1, \ldots, v_r \in V$ and $\omega^1, \ldots, \omega^s \in V^*$. (b) Show by induction that there is a unique isomorphism $\phi_s{}^r\colon T_s{}^r(V) \to T_{r+s}^0(V)$ for all nonnegative integers r, s such that $\phi_s{}^r(v_1 \otimes \cdots \otimes v_r \otimes \omega^1 \otimes \cdots \otimes \omega^s) = \phi(v_1) \otimes \cdots \otimes \phi(v_r) \otimes \omega^1 \otimes \cdots \otimes \omega^s$. (c) Suppose p, q and r, s are nonnegative integers such that $p + q = r + s$. For $A \in T_q{}^p(V)$ and $B \in T_s{}^r(V)$, define: A is *ϕ-equivalent* to B (in symbols: $A \sim B$) iff $\phi_q{}^p(A) = \phi_s{}^r(B)$. Show that $\sim$ is an equivalence relation.

EXERCISE 0.0.15

Let V be a finite dimensional vector space and g be an inner product on V. (a) Show that $\phi\colon V \to V^*$ defined by $(\phi v)w = g(v, w) \, \forall v, w \in V$ is an isomorphism. We shall call this ϕ the *metric isomorphism* (*induced by* g). (b) Show that the map

$\hat{g}: V^* \times V^* \to \mathbb{R}$ defined by $\hat{g}(\omega, \omega') = g(\phi^{-1}\omega, \phi^{-1}\omega')$, $\forall \omega, \omega' \in V^*$, is an inner product on V^*. (c) Show that index g = index $\hat{g}$. (In particular, $\hat{g}$ is Lorentzian iff g is.) (d) Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of V with respect to g, and let $\{\varepsilon^1, \ldots, \varepsilon^N\}$ be its dual basis. Show that $\{\varepsilon^1, \ldots, \varepsilon^N\}$ is orthonormal with respect to $\hat{g}$. (This justifies the terminology of "orthonormal basis of 1-forms" introduced in Section 0.0.9.) (e) Show that $\hat{g}$, considered as a (2, 0) tensor, is ϕ-equivalent to g in the sense of Exercise 0.0.14(c). (f) Show that the element of V ϕ-equivalent to an $\omega \in V^*$ is given by $\hat{g}(\omega, \cdot)$.

Exercise 0.0.16

Let V be a finite dimensional vector space, $\psi: V \to V$ be a given isomorphism, and $\psi^*: V^* \to V^*$ be the adjoint isomorphism. Show that for all nonnegative integers r, s there is a unique extension of ψ to an isomorphism $\psi_s{}^r: T_s{}^r(V) \to T_s{}^r(V)$ such that $(\psi_s{}^r A)(\omega^1, \ldots, \omega^r, v_1, \ldots, v_s) = A(\psi^*\omega^1, \ldots, \psi^*\omega^r, \psi v_1, \ldots, \psi v_s) \forall A \in T_s{}^r(V)$, $\omega^1, \ldots, \omega^r \in V^*$, and $v_1, \ldots, v_s \in V$.

0.1 Physics background

0.1.1 General relativity

No well-defined current physical theory claims to model all nature; each intentionally neglects some effects. Roughly, general relativity is a model of nature, especially of gravity, that neglects quantum effects. Its central assumption is that space, time, and gravity are all aspects of a single entity, called spacetime, which is modelled by a 4-dimensional Lorentzian manifold. It analyzes spacetime, electromagnetism, matter, and their mutual influences. It is used mainly in the study of large-scale phenomena: dense stars, the universe, and so on.

Now in microphysics, gravity counts as a very minor effect. For example the electric repulsion between two electrons is believed to be more than 10^{40} times as large as their mutual gravitational attraction. But gravity is long range and cumulative. In the realm of stars and galaxies it can dominate. For example, the discovery of pulsars has now made it virtually certain that there are some stars that manage to resist total collapse caused by their own gravity only by a last-ditch effort, at a radius of perhaps 10 miles. For such stars, and for the universe as a whole, general relativity is the best available theory. It is also believed that there are stars for which gravity has triumphed completely, collapsing the star to a black hole. If so, general relativity will become very exciting during the next decade.

Since we are giving a mathematical exposition of general relativity, the basic postulates of this branch of physics are of necessity disguised as definitions. The key definitions are given in Sections 1.3.1, 3.3.1, 3.4.2, 3.5, 3.7.1, and 4.1.1. These definitions, not theorems, are central. Such definitions carry the connotation "nature is really somewhat like that," so they require more motivation than purely mathematical definitions. But we shall soft-pedal motivations. Genuine motivations cannot be given piecemeal; they