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NONLINEAR DYNAMICS IN OPTICAL COMPLEX SYSTEMS

Advances in Optoelectronics (ADOP)

PREFACE

In the beginning of this century, when there existed no computers in the world, Poincaré already imagined the existence of extremely complex trajectories of motions through his research on three-body problems and mentioned that "*trajectories are surprisingly complex and I do not intend to draw their pictures...*". It took 60 years until unpredictability of such complex motions is recognized through research on weather [1] and motion of stars in the galaxy [2]. These two discoveries initiated the research field of nonlinear dynamics and chaos in science, and they are referred to as Lorenz chaos [1] and Henon-Heiles chaos [2]. Following these pioneering works, three universal routes to chaos, i.e., quasiperiodicity [3], period-doubling cascade [4] and intermittency [5], were discovered by early 80's. In accordance with these theoretical and mathematical progresses, nonlinear dynamics and chaos have been demonstrated experimentally in fields as diverse as chemistry, hydrodynamics, solid-state devices, biology, celestial mechanics, optics and so on. Rapid advancements in this field of nonlinear dynamics and chaos led to several discoveries of self-similarity laws and quantitive characterization methodology of chaotic motions (e.g., strange attractors). More recently, the research towards *coping with chaos* [6] utilizing the basic knowledge of the theory of chaos to achieve some practical goal, such as prediction, control and communication, has been demonstrated successfully.

Nonlinear dynamics and chaos in optics has a significant conceptual meaning, because fundamental models of nonlinear optical systems, which are derived from well-established Maxwell-Bloch equations, possess inherent instability leading to chaos. These models provide promising prototypes for investigating complex dynamical behaviors in strong connection with experimental demonstrations. The research on nonlinear optical dynamics forms the two poles together with the current topics in quantum optics such as squeezing, cavity quantum electrodynamics, laser cooling, Bose-Eistein condensation, quantum nondemolition measurement, so on and so forth, which present us a variety of microscopic quantum mechanical world.

Research on chaos in optics was triggered by the dicovery of the simple mathematical correspondence existing between Lorenz equations and Maxwell-Bloch laser equations with three variables by Haken in 1975 [7]. In early studies of solid-state lasers, period-doubling and chaotic spiking were already observed in a deeply modulated Nd:YAG laser and chaotic spiking oscillation was numerically observed in 1970 [8], although there was no concept of *chaos* in the field of Quantum Electronics. The proposal of Ikeda map in nonlinear passive optical resonators in 1979 [9], which exhibits a Feigenbaum's period-doubling bifurcation [4], observations of chaos in widely used practical laser diodes [10] and much more have accelerated the research of chaos in optics.

Generally speaking, the understanding of fundamental properties of chaos in small degrees of freedom systems such as Lorenz chaos has reached the period of maturity and interests are considered to shift towards dynamics of nonlinear systems with large degrees of freedom. Such systems are sometimes referred to as "complex systems" recently. The research along this line would give birth to new concepts and methodology in the next century in the process of systematic studies of complex systems which are far from the traditional condensed-matter physics which focuses on individual materials, mesoscopic and microscopic systems. Indeed, through our research of complex systems on the stage of quantum optics in recent years, the following generic properties have been recognized:

1. Complex systems are self-organized to preserve orders resulting from their nonlinearity, e.g., *vanishing gain circulation rule, antiphase dynamics, winner-takes-all dynamics and antiphase periodic states, universal power spectra relation, nonstationary chaos, etc.*
2. Complex systems self-create a variety of dynamic patterns, e.g., *chaotic itinerancy, self-formation of easy switching paths, majority-ruling switching, spot dancing, mode hopping, grouping chaos and cooperative synchronization, etc.*
3. Complex systems aquire various cooperative functions which are qualitatively different from individual elements, e.g., *domino dynamics, spatial chaos and felexible memory, factorial dynamic pattern memory, controlled switching-path formation and factorial pattern generation, parametric "linear" response, etc.*

This Monograph entitled "Nonlinear Dynamics in Optical Complex Systems," summarizes systematically our work on nonlinear dynamics and chaos in optics which has been done in these past 10 years at NTT Basic Research Laboratories, Université Libre de Bruxelles and Tokai University, focussing on nonlinear dynamics and cooperative functions in collective optical systems with large degrees of freedom. For this purpose, detailed derivations of traditional equations are spared to a certain extent and only key messages are included. The author would be more than happy if the readers could be stimulated by some generic nature in complex systems mentioned above.

The author is indebted to Profs. T. Kamiya and M. Ohtsu for providing him an opportunity of publishing this Monograph. He also thanks Prof. K. Ikeda of Ritsumeikan University, Prof. P. Mandel of Université Libre de Bruxelles (Belgium) and Prof. J.-L. Chern of National Cheng Kung University (Taiwan). Most of this book is written on the basis of articles co-authored by these collaborators. In particular, Sections 1.2 and 1.3 have been completed referring to discussions with Prof. Ikeda.

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Chapter 1

PROLOGUE TO NONLINEAR DYNAMICS IN OPTICAL COMPLEX SYSTEMS

In this chapter, fundamental model equations which describe the light-matter interaction are summarized. Then, several key optical systems, which initiated the research on nonlinear dynamics and chaos in optical systems, and their essential properties are reviewed as the basis for "optical complex systems" described in the following chapters.

Before discussing a variety of optical instabilities, it should be pointed out that there exist three characteristic frequencies which appear when we consider dynamics resulting from the light-matter interaction. It is interesting to note that instabilities associated with these frequencies appear under appropriate conditions.

These three frequencies are: (1) the relaxation oscillation frequency, (2) the longitudinal mode spacing frequency and (3) the Rabi precession frequency. Relaxation oscillations refer to oscillations resulting from the interplay between the photon number and the population inversion in lasers, such as solid-state lasers, CO_2 laser, and semiconductor lasers, in which the transverse relaxation rate is extremely large as compared with the longitudinal relaxation rate, i.e., $\gamma_\perp \gg \gamma_\parallel$. The longitudinal cavity mode spacing frequency corresponds to the frequency difference between longitudinal modes which resonate in the cavity. This frequency plays an important role for instabilities in the passive nonlinear resonator and corresponds to the fundamental frequencies in multimode oscillations as well. Finally, Rabi precession refers to coherent oscillations of material fields (i.e., polarizations) which take place under the applied electric field in far-infrared lasers.

A schematic diagram of time scales of optical instabilities under typical oscillation conditions is depicted in Fig. 1.1, showing the relation between characteristic frequencies and three relaxation rates, κ, $\gamma_\perp$ and $\gamma_\parallel$, which characterize the optical resonator and the nonlinear medium. Here, κ is

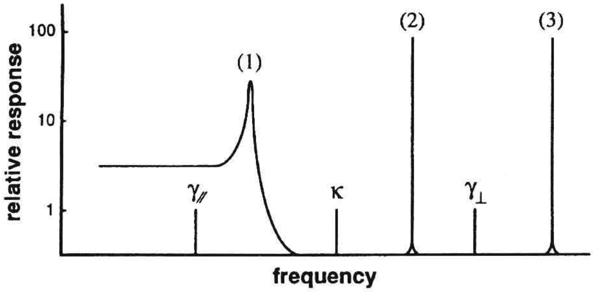

Figure 1.1: Schematic diagram of laser instabilities. Shown is the output modulation (peak intensity over average intensity) versus frequency of cavity loss modulation.

the damping rate of the optical cavity.

In section **1.1**, fundamental semiclassical equations which describe the light-matter interaction are described. Sections **1.2** and **1.3** describe laser instabilities and chaos associated with Rabi precession and the effect of inhomogeneous broadening on laser dynamics. Section **1.4** discusses instabilities and chaos related to relaxation oscillations. Finally, instabilities and chaos in passive nonlinear resonators are reviewed in section **1.5**.

1.1 Fundamental Semiclassical Equations

We consider atoms interacting with an "intense" electromagnetic field for which the quantization is not necessary. Starting from the Schrődinger equation, the following dynamical equations describing the light-matter interaction in two-level systems are derived in terms of density matrix elements ρ_{ij}:

$$d\rho_{12}/dt = (i\omega_0 - \gamma_\perp)\rho_{12} - i(\mu_{12}/h)E\Delta\rho, \tag{1.1}$$

$$d\Delta\rho/dt = i(4\pi\mu_{12}/h)E(\rho_{12} - \rho_{12}^*) + \gamma_{\|}(\Delta\rho_0 - \Delta\rho), \tag{1.2}$$

where $h\omega_0/2\pi = W_2 - W_1$ (W_i is the energy level, ω_0 is the atomic resonance frequency), μ_{12} is the electric dipole moment, $\Delta\rho = \rho_{22} - \rho_{11}$ and $\Delta\rho_0 = \rho_{22}^{(0)} - \rho_{11}^{(0)}$.

From Maxwell equations, on the other hand, wave equations for the electromagnetic field $E(\mathbf{r}, t)$ and material polarization $P(\mathbf{r}, t)$ in the cavity

$$E(\mathbf{r}, t) = \sum_n \tilde{E}_n(t)U_n(\mathbf{r})e^{i\omega_n t} + c.c., \tag{1.3}$$

$$P(\mathbf{r}, t) = \sum_n \tilde{P}_n(t)U_n(\mathbf{r})e^{i\omega_n t} + c.c., \tag{1.4}$$

are given by

$$d\tilde{E}_n/dt + [\kappa_n + i(\Omega_n - \omega_n)]\tilde{E}_n = i(\omega_n/2\epsilon_0)\tilde{P}_n, \quad (1.5)$$

assuming rotating-wave and slowly-varying envelope amplitude (SVEA) approximations. Here, κ_n is the damping rate, Ω_n is the cavity resonance frequency, ω_n is the oscillation frequency of the n-th mode and ϵ_0 is the dielectric constant.

The macroscopic material polarization and population inversion are expressed by $P = N\rho_{12}\mu_{12} + c.c.$ and $\Delta N = N\Delta\rho$ is the population inversion density, where N is the density of atoms. Assume a single-mode oscillation, then the following Maxwell-Bloch laser equations for the complex field amplitude $\tilde{E}$, the complex polarization amplitude $\tilde{\rho}$ oscillating at the frequency ω and $\Delta\rho$ are derived from Eqs. (1.1), (1.2) and (1.5)

$$d\tilde{E}/dt = -\kappa\tilde{E} - (N\mu\omega/2\epsilon_0)\tilde{\rho}, \quad (1.6)$$

$$d\tilde{\rho}/dt = [i(\omega - \omega_0) - \gamma_\perp]\tilde{\rho} - (2\pi\mu/h)\tilde{E}\tilde{\Delta}\rho, \quad (1.7)$$

$$d\tilde{\Delta}\rho/dt = \gamma_\parallel(\Delta\rho_0 - \tilde{\Delta}\rho) + (\pi\mu/h)(\tilde{E}\tilde{\rho}^* + \tilde{E}^*\tilde{\rho}), \quad (1.8)$$

where subscripts are omitted.

In the stationary state, from the relation $P = \epsilon_0\chi E$, the complex electric susceptibility χ is derived from the Bloch equations (1.6)–(1.8) as

$$\chi \equiv \chi' - i\chi'' = \frac{2\pi N\Delta\rho_0|\mu|^2[(\omega_0 - \omega) - i\gamma_\perp]}{\epsilon_0 h[(\omega_0 - \omega)^2 + \gamma_\perp^2 + (2\pi\mu E/h)^2(\gamma_\perp/\gamma_\parallel)]}. \quad (1.9)$$

Here, the real part expresses the dispersion (refractive index) profile and the imaginary part expresses the light amplification (absorption) profile. χ' and χ'' obey Kramers-Kronig relationship, where the term $(2\pi\mu E/h)^2(\gamma_\perp/\gamma_\parallel)$ is a light-intensity dependent $\chi^{(3)}$ nonlinearity of the refractive index and gain (absorption).

1.2 Homogeneously-Broadened Single-Mode Laser

1.2.1 Stationary states and linear stability analysis

Let us consider the case that the oscillation frequency ω is tuned to the atomic transition frequency ω_0, i.e., $\omega = \Omega = \omega_0$. Here, we introduce a new variable $F = -(2\pi\mu/h)\tilde{E}$ and notations are changed as $\tilde{\rho} \to \rho$ and $\tilde{\Delta}\rho \to w$. In this case, Eqs. (1.6)–(1.8) are written as

$$dF/dt = -\kappa F + s^2\rho, \quad (1.10)$$

$$d\rho/dt = -\gamma_\perp\rho + Fw, \quad (1.11)$$

$$dw/dt = \gamma_{\|}(w_0 - w) + F\rho, \tag{1.12}$$

where $s^2 \equiv N\omega|\mu|^2/2\epsilon_0$.

We abbreviate the set of 3 variables (F, ρ, w) of the Maxwell-Bloch equations (1.10)–(1.12) with vector $\mathbf{x}$ and let Eqs. (1.10)–(1.12) be written formally by

$$d\mathbf{x}/dt = \mathbf{F}(\mathbf{x}) \tag{1.13}$$

$\mathbf{F}(\mathbf{x})$ defines the flow vector at the point $\mathbf{x}$=(F, ρ, w) in the 3-dimensional phase space whose components are the r.h.s. of Eqs. (1.10)–(1.12). Stationary solutions, i.e., fixed point $\mathbf{x}_s$ is the point which satisfies $d\mathbf{x}_s/dt = 0$, i.e.,

$$\mathbf{F}(\mathbf{x_s}) = \mathbf{O}$$

The problem is the stability of $\mathbf{x}_s$. The infinitesimal deviation $\delta\mathbf{x} = (\delta F, \delta\rho, \delta\omega)$ around $\mathbf{x}_s$ obeys the following linearized equation of motion

$$d\delta\mathbf{x}/dt = (\partial\mathbf{F}(\mathbf{x}_s)/\partial\mathbf{x})\delta\mathbf{x} \tag{1.14}$$

$\partial\mathbf{F}(\mathbf{x}_s)/\partial\mathbf{x}$ represent a 3×3 Jacobian matrix with (i,j) components given by $\partial F_i(\mathbf{x_s})/\partial\mathbf{x_j}$. The stability analysis of the model equations (1.10)–(1.12), is quite simple, and is left as an exercise for the readers. Only the final results are shown below:

(1) If the linear gain $\alpha_0 = s^2 w_0/\gamma_\perp$ is smaller than the threshold value $\alpha_{th}^{(1)} = \kappa$, only the trivial non-lasing solution,

$$\mathbf{O} : (F_s, \rho_s, w_s) = (0, 0, w_0) \tag{1.15}$$

exists as a stable fixed point solution.

(2) If α_0 exceeds the first thresheld $\alpha_{th}^{(1)}$, $\mathbf{O}$ becomes unstable, and the lasing solution

$$\mathbf{L}^{\pm} : (F_s, \rho_s, w_s) = (\pm\sqrt{\gamma_{\|}\gamma_\perp(\frac{\alpha_0}{\sigma} - 1)}, \pm\frac{\kappa}{s^2}\sqrt{\gamma_{\|}\gamma_\perp(\frac{\alpha_0}{\sigma} - 1)}, \frac{\gamma_\perp\kappa}{s^2}) \tag{1.16}$$

appeas as a new stable fixed points. At $\alpha_0 = \alpha_{th}^{(1)}$, the stability exponents at $\mathbf{O}$ is

$$\lambda_1 = 0, \quad \lambda_2 = -\gamma_{\|} \quad \lambda_3 = -(\kappa + \gamma_\perp), \tag{1.17}$$

and λ_1 becomes positive at $\alpha_0 > \alpha_{th}^{(1)}$.

(3) If α_0 is increased further and under the condition

$$\kappa > \gamma_{\|} + \gamma_\perp, \tag{1.18}$$

even the fixed points $\mathbf{L}^{\pm}$ become unstable when α_0 exceeds the "seccond" threshold. The ratio of $\alpha_{th}^{(2)}$ to $\alpha_{th}^{(1)}$ is given by

$$\alpha_{th}^{(2)}/\alpha_{th}^{(1)} = \frac{\kappa}{\gamma_\perp}(\frac{\kappa}{\gamma_\perp} + \frac{\gamma_\parallel}{\gamma_\perp} + 3)/(\frac{\kappa}{\gamma_\perp} - \frac{\gamma_\parallel}{\gamma_\perp} - 1). \tag{1.19}$$

and the stability exponents around $\mathbf{L}^{\pm}$ at $\alpha_0 = \alpha_{th}^{(2)}$ are given by,

$$\lambda_1 = -(\kappa + \gamma_\parallel + \gamma_\perp), \quad \lambda_{2,3} = \pm i\sqrt{2\kappa(\kappa + \kappa_\perp)(\kappa - \gamma_\parallel - \gamma_\perp)/\gamma_\perp} \tag{1.20}$$

For $\alpha_0 > \alpha_{th}^{(2)}$, the real part of $\lambda_{2,3}$ becomes positive, and a precession at frequency $Im(\lambda_{2,3})$ is excited. Consequently, the bifurcation phenomenon which occurs at $\alpha_0 = \alpha_{th}^{(2)}$ is a Hopf bifurcation.

The above descriptions are too much arithmetic. We therefore present a physical picture of the instability above the second threshold since the indicated phenomenon contains a typical aspect of laser instabilities.

For the sake of simplicity, let us consider the special case of $\gamma_\parallel = \gamma_\perp \equiv \gamma$, and introduce a complex variable Z representing the material field

$$Z = s^2(\rho - iw). \tag{1.21}$$

The linearized equation (1.14) is then written as

$$d\delta F/dt = -\kappa\delta F + (\delta Z + \delta Z^*)/2 \tag{1.22a}$$

$$d\delta Z/dt = -\gamma\delta Z + iZ_s\delta F + iF_s\delta Z \tag{1.22b}$$

Let us now consider the limit of large linear gain $\alpha_0 = s^2 w_0/\gamma_\perp$. The third term on the right side of equation (1.22b) becomes dominant. This means that the deviation δZ of the material field starts to oscillate as $\delta Z \sim \tilde{\delta Z} e^{iF_s t}$, where $\tilde{\delta Z}$ is the slowly varying part of δZ. This oscillation is the Rabi precession of the material field. The precession components of the material field, via the Maxwell equation (1.22a), induce the oscillation of electric field at the same frequency. Let $\delta F \sim \tilde{\delta F} e^{iF_s t}$, then $\tilde{\delta F} \sim \tilde{\delta Z}/2(iF_s + \kappa)$. In short,

$$\delta F \sim \frac{\delta Z}{2(iF_s + \kappa)}. \tag{1.23}$$

Note that the iF_s term resulting from the Rabi precession is included in the denominator. In the limit of $\alpha_0 \gg \kappa, i.e., F_s \to \infty$, this effect is significant and δF possesses only the out-of-phase component of δZ. In other words, induced electric fields are dispersive.