ALM 20

Advanced Lectures in Mathematics

Surveys in Geometric Analysis and Relativity

几何分析与相对论

Editors: Hubert L. Bray · William P. Minicozzi II

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ADVANCED LECTURES IN MATHEMATICS

Dedicated to Richard Schoen on the occasion of his sixtieth birthday

Preface

This volume of 23 survey articles is dedicated to Richard M. Schoen on the occasion of his 60th birthday in recognition of his many important contributions as a leading researcher in geometric analysis and general relativity. We also thank him for the equally important roles he has played as mentor, colleague, collaborator, and friend.

Rick Schoen was born on October 23, 1950 in Celina, Ohio. In 1972 he graduated summa cum laude from the University of Dayton and received an NSF Graduate Fellowship. In March 1977, Rick received his Ph.D. from Stanford University under the direction of Leon Simon and Shing-Tung Yau, and soon after received a Sloan Postdoctoral Fellowship. His early work was on minimal surfaces and harmonic maps. By the time that Rick received his Ph.D., he had already proven major results, including his 1975 curvature estimates paper with Simon and Yau.

In the late 1970's, Schoen and Yau developed new tools to study the topological implications of positive scalar curvature. This work grew out of their study of stable minimal surfaces, eventually leading to their proof of the positive mass theorem in 1979. All together, their work was impressive for the way it connected neighboring fields, first using analysis to understand geometry, and then using geometry to understand physics.

In the early 1980's, Rick published a number of fundamental papers on minimal surfaces and harmonic maps. His work on minimal surfaces includes an influential Bernstein theorem for stable minimal surfaces with Doris Fischer-Colbrie. Rick met his future wife Doris in Berkeley, where Doris received her Ph.D. in 1978. They have two children, Alan and Lucy, seen in the photographs in this book, both of whom graduated from Stanford.

Other works from the early 1980's include an extremely useful curvature estimate for stable surfaces, a uniqueness theorem for the catenoid, and a partial regularity theory for stable hypersurfaces in high dimensions with Leon Simon. In 1982, Rick and Karen Uhlenbeck proved the partial regularity of energy minimizing harmonic maps. In 1983, Rick was awarded the very prestigious MacArthur Prize Fellowship.

Rick is also very well known for his celebrated solution to the remaining cases of the Yamabe problem in 1984, this time using a theorem from physics, namely the positive mass theorem, to solve a famous problem in geometry. The resulting fundamental theorem in geometry, that every smooth Riemannian metric on a closed manifold admits a conformal metric of constant scalar curvature, had been open since the 60's. This work was cited in 1989 when Rick received the Bocher prize of the American Mathematical Society. His work on scalar curvature at this time set the direction for the field for the next 25 years.

Rick was elected to the American Academy of Arts and Sciences in 1988 and the National Academy of Sciences in 1991. He has been a Fellow of the American Association for the Advancement of Science since 1995 and won a Guggenheim Fellowship in 1996.

Starting around 1990, Rick began two major programs. The first was to develop a theory of harmonic maps with singular targets, starting with a joint paper with Mikhail Gromov where they used harmonic maps to establish p-adic superrigidity for lattices in groups of rank one. In a series of papers, Rick and Nick Korevaar laid the foundations for a general theory of mappings to NPC spaces, established the basic existence and regularity results, and applied their theory to settle problems in a number of areas of mathematics. The second big program was a variational theory of Lagrangian submanifolds, including the existence and regularity theory, done in a series of papers with Jon Wolfson.

Over the last decade, Rick has continued to make major contributions to geometric analysis and general relativity. Among other results in general relativity, Rick has made fundamental contributions to the constraint equations (with Corvino and others) which dictate the range of possible initial conditions for a spacetime and proved theorems on the topology of higher dimensional black holes (with Galloway). In geometric analysis, he has several important results with Simon Brendle on Ricci flow, including the proof of the differentiable sphere theorem, as well as a compactness theorem for the Yamabe equation with Marcus Khuri and Fernando Marques.

Rick has written 2 books and roughly 80 papers and has solved an impressively wide variety of major problems and conjectures. He has supervised 35 students and counting, and he has hosted many postdocs. Even with his great success, Rick is still one of the hardest working people in mathematics, giving us all the distinct impression that he must love it. His impact on mathematics, both in terms of his ideas and the example he sets, continues to be tremendous.

We would like to thank all of the authors for their contributions, the publishers Lizhen Ji and Liping Wang for their help, as well as Jaigyoung Choe, Michael Eichmair, John Rawnsley, Peter Topping, and Doris Fischer-Colbrie for contributing photographs. We hope you enjoy reading the beautiful survey articles included in this volume as much as we have enjoyed helping to put it all together.

Hubert L. Bray and William P. Minicozzi II
April 20, 2011

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On the Positive Mass, Penrose, and ZAS Inequalities in General Dimension

Hubert L. Bray*

Abstract

After a detailed introduction including new examples, we give an exposition focusing on the Riemannian cases of the positive mass, Penrose, and ZAS inequalities of general relativity, in general dimension.

2000 Mathematics Subject Classification: 53C80, 53C24, 53C44.
Keywords and Phrases: General relativity, Scalar curvature, Black holes, Minimal surfaces, Zero area singularities, ZAS, Positive mass inequality, Penrose inequality, ZAS inequality.

1 Dedication

It is an honor and a pleasure to contribute a paper to this volume celebrating Richard Schoen's 60th birthday (which doesn't seem possible since Rick is so much better at every sport than the author). Rick's contribution to mathematics continues to be tremendous and is growing at an impressive rate, including his original research, collaborations with others, and the many students whom he has so expertly supervised. As successful as he has been for so many years, his continuing hard work and dedication reveals his genuine passion and love for mathematics. He is a role model for us all.

The ideas and directions discussed in this paper were in large part inspired by the positive mass inequality, proved in dimensions less than eight in Rick's famous joint works with S.-T. Yau [37, 38, 36, 35]. The positive mass inequality is a beautiful geometric statement about scalar curvature which can equivalently be understood as the fundamental result in general relativity that positive energy densities in a spacetime imply that the total mass of the spacetime is also positive. In this paper we will discuss the Riemannian cases of the positive mass inequality, the Penrose inequality, and the ZAS (zero area singularity) inequality, describing how they are closely related in Section 3. We apologize in advance for not making this a comprehensive survey of every interesting result in these areas, although we do recommend [26] as a very nice survey of the Penrose inequality. But first,

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in Section 2 we begin with a mixture of well-known facts and interesting new examples to motivate our later discussion.

2 Introduction

In this section we describe some of the ideas, motivations, and definitions that are central to general relativity, in general dimension. We also discuss intuition and include some new examples due to Lam [21, 22] of manifolds where the Riemannian positive mass inequality and the Riemannian Penrose inequality can be proved directly.

Broadly defined, general relativity encompasses any description of the universe as a smooth manifold with a Lorentzian metric, called a spacetime, of signature $(1, n)$, where usually $n = 3$. Then the Einstein equation

$$G = (n-1)\omega_{n-1}T \tag{1}$$

can be taken to be the definition of the stress energy tensor T, where ω_{n-1} is the measure of the unit $(n-1)$ sphere. Thus, in dimension three, $G = 8\pi T$. The Einstein curvature tensor $G = \text{Ric} - \frac{1}{2}R \cdot g$ has zero divergence by the second Bianchi identity. Furthermore, $T(\nu_1, \nu_2)$ is defined to be the amount of energy traveling in the unit direction ν_1 as observed by someone going in the unit direction ν_2. We comment that the Einstein equation can be derived from an action principle based on the Einstein-Hilbert action in the vacuum case, but we will take the Einstein equation as our starting point. The zero divergence property of G (which is a consequence of the action principle) may be interpreted as a local conservation property for the stress energy tensor T.

2.1 The Schwarzschild Spacetimes

The vacuum case of general relativity is $G = 0$ which, for $n \geq 2$, implies that $\text{Ric} = 0$. Unlike Newtonian physics which is trivial in the vacuum case, general relativity is highly nontrivial in the vacuum case when $n \geq 3$. That is, there exist many solutions to $G = 0$ other than the Minkowski spacetime metric

$$-dt \otimes dt + dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n, \tag{2}$$

which include solutions describing gravitational waves which propagate at the speed of light. However, if we restrict our attention to spacetimes which are static and spatially spherically symmetric, then it is an exercise to show that for $n \geq 3$ there is precisely a one parameter family of vacuum solutions, called the Schwarzschild spacetimes. Explicitly, these spacetimes are

$$-\left(\frac{1 - \frac{k}{r^{n-2}}}{1 + \frac{k}{r^{n-2}}}\right)^2 dt \otimes dt + \left(1 + \frac{k}{r^{n-2}}\right)^{\frac{4}{n-2}} \left(dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n\right), \tag{3}$$

for $r > |k|^{1/(n-2)}$, where $r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$.