PERSPECTIVES IN LOGIC

Jeffrey Paris
Alena Vencovská

PURE INDUCTIVE LOGIC

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ASL
ASSOCIATION FOR SYMBOLIC LOGIC

CAMBRIDGE
UNIVERSITY PRESS

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UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107042308

Association for Symbolic Logic
Richard Shore, Publisher
Department of Mathematics, Cornell University, Ithaca, NY 14853
http://www.aslonline.org

First published 2015

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data

Paris, J. B. (Jeff B.), author.
Pure inductive logic / Jeffrey Paris, University of Manchester;
Alena Vencovská, University of Manchester.
pages cm. – (Perspectives in logic)
Includes bibliographical references and index.
ISBN 978-1-107-04230-8 (Hardback)
1. Induction (Logic) I. Vencovská, Alena, author. II. Title.
BC91.P37 2015
161–dc23 2015004107

ISBN 978-1-107-04230-8 Hardback

Pure Inductive Logic

Pure inductive logic is the study of rational probability treated as a branch of mathematical logic. This monograph, the first devoted to this approach, brings together the key results from the past seventy years, plus the main contributions of the authors and their collaborators over the last decade, to present a comprehensive account of the discipline within a single unified context. The exposition is structured around the traditional bases of rationality, such as avoiding Dutch Books, respecting symmetry and ignoring irrelevant information. The authors uncover further rationality concepts, both in the unary and in the newly emerging polyadic languages, such as conformity, spectrum exchangeability, similarity and language invariance. For logicians with a mathematical grounding, this book provides a complete self-contained course on the subject, taking the reader from the basics up to the most recent developments. It is also a useful reference for a wider audience from philosophy and computer science.

JEFFREY PARIS is a professor in the School of Mathematics at the University of Manchester. His research interests lie in mathematical logic, particularly set theory, models of arithmetic and non-standard logics. In 1983 he was awarded the London Mathematical Society's Junior Whitehead Prize and in 1999 was elected a Fellow of the British Academy in the Philosophy Section. He is the author of *The Uncertain Reasoner's Companion* (Cambridge University Press, 1995).

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PERSPECTIVES IN LOGIC

The *Perspectives in Logic* series publishes substantial, high-quality books whose central theme lies in any area or aspect of logic. Books that present new material not now available in book form are particularly welcome. The series ranges from introductory texts suitable for beginning graduate courses to specialized monographs at the frontiers of research. Each book offers an illuminating perspective for its intended audience.

The series has its origins in the *old Perspectives in Mathematical Logic* series edited by the Ω-Group for "Mathematische Logik" of the Heidelberger Akademie der Wissenschaften, whose beginnings date back to the 1960s. The Association for Symbolic Logic has assumed editorial responsibility for the series and changed its name to reflect its interest in books that span the full range of disciplines in which logic plays an important role.

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PREFACE

We have been motivated to write this monograph by a wish to provide an introduction to an emerging area of Uncertain Reasoning, Pure Inductive Logic. Starting with John Maynard Keynes's 'Treatise on Probability' in 1921 there have been many books on, or touching on, Inductive Logic as we now understand it, but to our knowledge this one is the first to treat and develop that area as a discipline within Mathematical Logic, as opposed to within Philosophy. It is timely to do so now because of the subject's recent rapid development and because, by collecting together in one volume what we perceive to be the main results to date in the area, we would hope to encourage the subject's continued growth and good health.

This is primarily a text aimed at mathematical logicians, or philosophical logicians with a good grasp of Mathematics. However the subject itself gains its direction and motivation from considerations of rational reasoning which very much lie within the province of Philosophy, and it should also be relevant to Artificial Intelligence. For this reason we would hope that even at a somewhat more superficial and circumspect reading it will have something worthwhile to say to that wider community, and certainly the link must be maintained if the subject is not to degenerate into simply doing more mathematics just for the sake of it. Having said that however we will not be lingering particularly on the more philosophical aspects and considerations nor will we speculate about possible application within AI. Rather we will mostly be proving theorems and leaving the reader to interpret them in her or his lights.

This monograph is divided into three parts. In the first we have tried to give a rather gentle introduction and this should be reasonably accessible to quite a wide audience. In the second part, which deals with 'classical' Unary Pure Inductive Logic, the mathematics required is a little more demanding, with more being left for the reader to fill in, and this trend continues in the final, 'post classical', part on Polyadic Pure Inductive Logic. In presenting the results we have tried to keep the material reasonably self contained though on a few occasions it has proved necessary to import a 'big theorem' from outside; this happens in particular in the third part where Nonstandard Analysis is used. In all such cases however we provide source references.

The chapters have been arranged, hardly surprisingly, so that they can be read straight through from start to finish. However an alternative approach would be to familiarize oneself with Part 1, though even there Chapter 5 on the Dutch Book could be omitted, and then simply dip in to selected later chapters, only referring back to earlier material as subsequently proved necessary. To aid this approach we have included at the end a rather extensive list of symbols and abbreviations used in the text and have generally tried to outline the contents of a chapter in its first paragraph or so.

Much of the material we will present has previously appeared in journals or books. What we have endeavoured is to collect and present it in a unified notation and fashion, in so doing making it, we hope, more easily accessible to a wider audience. Whilst most of the later chapters derive from research of ourselves and our collaborators, and here we should make a special mention of Jürgen Landes and Chris Nix, this monograph naturally owes a debt in its first two parts to the seminal contributions of the founding fathers of the area, in particular W.E. Johnson, Rudolf Carnap, and more recently Haim Gaifman and Jakko Hintikka. Doubtless there are many other results and attributions we should also have included and for this we can only apologize and, to quote Dr Samuel Johnson, blame 'pure ignorance'.

We would also like to thank Jon Williamson, Teddy Groves, our colleague George Wilmers and our recent research students, Martin Adamčik, Alex Hill, Hykel Hosni, Elizabeth Howarth, Malte Kließ, Jürgen Landes, Soroush Rafiee Rad and Tahel Ronel for diligently reading earlier drafts and spotting errors. Any that remain are, of course, entirely our fault.

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Part 1

THE BASICS

Chapter 1

INTRODUCTION TO PURE INDUCTIVE LOGIC

Before a cricket match can begin the tradition is that the umpire tosses a coin and one of the captains calls, heads or tails, whilst the coin is in the air. If the captain gets it right s/he chooses which side opens the batting. There never seems to be an issue as to which captain actually makes this call (otherwise we would have to toss a coin and make a call to decide who makes the call, and in turn toss a coin and make a call to decide who makes that call and so on) since it seems clear that this procedure is fair. In other words both captains are giving equal probability to the coin landing heads as to it landing tails no matter which of them calls it. The obvious explanation for this is that both captains are, subconsciously perhaps, appealing to the *symmetry* of the situation.

At the same time they are, it seems, also tacitly making the assumption that all the other information they possess about the situation, for example the weather, the gender of the referee, even past successes at coin calling, is *irrelevant*, at least if it doesn't involve some specific knowledge about this particular coin or the umpires's ability to influence the outcome. Of course if we knew that on the last 8 occasions on which this particular umpire had tossed up this same coin the result had been heads we might well consider that that *was relevant*.

Forming beliefs, or subjective probabilities, in this way by considering symmetry, irrelevance, relevance, can be thought of as *logical* or *rational* inference. This is something different from statistical inference. The perceived fairness of the coin toss is clearly not based on the captains' knowledge of a long run of past tosses by the umpire which have favoured heads close to half the time. Indeed it is conceivable that this long run frequency might not give an average of close to half heads, maybe this coin is, contrary to appearances, biased. Nevertheless even if the captains knew that the coin was biased, provided that they also knew that the caller was not privy to which side of the coin was favoured, they would surely still consider the process as fair.

This illustrates another feature of probabilities that are inferred on logical grounds: they certainly need not agree with the long term frequency probability, if this even exists, and of course in many situations in which we form subjective probabilities no such probability does exist; for example when assigning odds in a horse race.

The aim of this monograph is to investigate this process of assigning logical, as opposed to statistical, probabilities by attempting to formulate the underlying notions, such as symmetry, irrelevance, relevance on which they appear to depend. Much has already been written by philosophers on these matters and doubtless much still remains to be said. Our approach here however will be that of mathematical, rather than philosophical, logicians. So instead of spending a significant time discussing these notions at length in the context of specific examples we shall largely consider ways in which they might be given a purely mathematical formulation and then devote our main effort to considering the mathematical and logical consequences which ensue.

In this way then we are proposing, or at least reviving since Rudolf Carnap had already introduced the notion in [14], an area of Mathematical Logic, *Pure Inductive Logic*, PIL for short.[1] It is not Philosophy as such but there are close connections. Firstly most of the logical, aka rational, principles we consider are motivated by philosophical considerations, frequently having an already established presence in the literature within that subject. Secondly we would hope that the mathematical results included here may feed back and contribute to the continuing debates within Philosophy itself, if only by clarifying that *if* you subscribe to A, B, C *then* you must, by dint of mathematical proof, accept D.

There is a parallel here with Set Theory. In that case we propose axioms based on our intuitions concerning the nature of sets and then investigate their consequences. These axioms have philosophical content and considering this is part of the picture but so also is drawing out their mathematical relationships and consequences. And as we go deeper into the subject we are led to propose or investigate axioms which initially might not have entered our minds, not least because we may well not have possessed the language or notions to even express them. And at the end of the day most of us would like to think that discoveries in Set Theory were telling us something about the universe of sets, or at least about possible universes of sets, and thus feeding back into the philosophical debate (and not simply generating yet more mathematics 'because it is there'!). Hopefully Pure Inductive Logic, PIL, will similarly tell us something about the universe of uncertain reasoning.

As far as the origins of PIL are concerned, whilst one may hark back to Keynes, Mill and even as far as Aristotle, in the more recent history of logical probability as we see it, W.E. Johnson's posthumous 1932 paper [58] in *Mind* was the first important contribution in the general spirit of what we

[1] For a gentle introduction to PIL see also [100].

are proposing here. It contains an initial assertion of mathematical conditions capturing intuitively attractive principles of uncertain reasoning and a derivation from these of what subsequently became known as Carnap's Continuum of Inductive Methods. Independently Carnap was to follow a similar line of enquiry in his [9], [12], which he developed further with [13], [15], [16], [17] into the subject he dubbed 'Inductive Logic'. Already in 1946 however N. Goodman's so called 'grue' paradox, see [35], [36] (to which Carnap responded with [10], [11]) threatened to capsize the whole venture by calling into question the very possibility of a purely logical basis for inductive inference[2]. Notwithstanding Carnap maintained his commitment to the idea of an Inductive Logic till his death in 1970 and to the present day his vision encourages a small but dedicated band of largely philosopher logicians to continue the original venture in a similar spirit, albeit in the ubiquitous shadow of 'grue'.

From the point of view of this text however 'grue' is no paradox at all, it is just the result of failing to make explicit all the assumptions that were being used. There is no isomorphism between premises involving grue and green (a point we will touch on again later in the footnote on page 177) because we have different background knowledge concerning grue and green etc. and it is precisely this knowledge which the paradox subsequently uses to announce a contradiction.[3] Indeed in his initial response to 'grue' Carnap had also stressed the importance of having all the assumptions up front from the start, what he called the 'Principle of Total Evidence', see [10, p138], [12, p211], known earlier as 'Bernoulli's Maxim', see [6, footnote 1, p215], [65, p76, p313].

Even so, 'grue' is relevant to this monograph in that it highlights a divergence within Carnap's Inductive Logic as to its focus or subject matter between *Pure Inductive Logic*, which is our interest in this monograph, and *Applied Inductive Logic*, which is the practical concern of many philosophers. The former was already outlined by Carnap in [14]; it aims to

[2]For the reader unfamiliar with this 'paradox' here is a pared down mathematician's version: Let *grue* stand for 'green before the 1st of next month, blue after'. Now consider the following statements:

All the emeralds I have ever seen have been green, so I should give high probability that any emerald I see next month will be green.

All the emeralds I have ever seen have been grue, so I should give high probability that any emerald I see next month will be grue.

The conclusion that advocates of this 'paradox' would have us conclude is that Carnap's hope of determining such probabilities by purely logical or rational considerations cannot succeed. For here are 'isomorphic' premises with different (contradictory even) conclusions so the conclusion cannot simply be a logical function of the available information.

[3]For example we learnt in school that emeralds are green and never heard anything about this possibly changing at some future date. In contrast if we had been talking here about UK Road Fund Licence discs changing to a new colour next January 1st there would have been a 'paradox' for the contrary reason!

study formal systems in the mathematical sense, devoid of explicit interpretation. Assumptions must be stated within the formal language and conclusions drawn only on the basis of explicitly given rules. On the other hand Applied Inductive Logic is intended as a tool, in particular, to sanction degrees of confirmation, within particular contexts. The language therein is interpreted and so carries with it knowledge and assumptions. What Goodman's Paradox points out is that applied in this fashion the conclusions of any such logic may be language dependent (see [136] for a straightforward amplification of this point), a stumbling block which has spawned a considerable literature, for example [131], [137], and which, within PIL, we thankfully avoid. In short then we might draw a parallel here with the aims and methods of Pure Mathematics as opposed to those of Applied Mathematics.

In the latter we begin with an immensely complicated real world situation, cut it down to manageable size by choosing what we consider to be the relevant variables and the relevant constraints, so ignoring a wealth of other information which we judge irrelevant, and then, *drawing on existing mathematical theories and apparatus*, we hopefully come up with some predictive or explicative formula. Similarly with Inductive Logic the applied arm has been largely concerned with proposing formulae in such contexts - prior probability functions, to provide answers. The value of these answers and the whole enterprize has been subject to near century long debate, some philosophers feeling that the project is fundamentally flawed. On the other hand it clearly finds new challenges with the advent of designing artificial reasoning agents. Be it as it may, PIL is not out to *prescribe* priors. Rather it is an investigation into the various notions of 'rationality' in the context of forming beliefs as probabilities. It is in this foundational sense that we hope this monograph may be of interest to philosophers and to the Artificial Intelligence community. Similarly to other mathematical theories, we would hope that it would serve to aid any researcher contemplating actual problems related to rational reasoning.

A rough plan of this monograph is as follows. In the early chapters we shall introduce the basic notation and general results about probability functions for predicate languages, as well as explaining what we see as the most attractive justification (de Finetti's Dutch Book argument) for identifying degrees of belief with probability. We will then investigate principles based on considerations of symmetry, relevance, irrelevance and analogy, amongst others, for *Unary Pure Inductive Logic*, that is for predicate languages with only unary relation symbols. This was the context for Inductive Logic in which Carnap et al worked and with only a very few exceptions it remained so until the end of the 20th century. In the second half of this monograph we will enlarge the framework to *Polyadic Pure Inductive Logic* and return to reconsider symmetry, relevance and irrelevance within this wider context.