Theory of
Partial Differential Equations

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Preface

This book is written in four modular parts intended as easy steps for the student. The intention here is to lead him from an elementary level to a level of modern analysis research. Thus the first pages of Part I are an explanation of the regular (classical) solutions of the second-order wave equation in two-space–time, while the latter pages of Part IV encompass a more or less complete analysis of the existence of $\mathscr{L}^p$-weak solutions for boundary-value problems for equations of elliptic–parabolic type expounded according to G. Fichera of the University of Rome. In the developing process, an effort is made to ensure that the student samples the extensive variety of mathematically conceivable boundary-value problems, even if their properties are not entirely satisfying once analyzed, and that he learns how to use these tools to elucidate phenomena of nature and technology.

The field is to some extent characterized by the fact that one rarely "solves" boundary-value problems in any acceptable sense of the word. Since computing plays an altogether inseparable role in approximating solutions to boundary-value problems, we present wherever possible a skeleton of the basic theoretical framework for the numerical analysis of several problems along with that of the theory of existence, uniqueness, and integral representations. Where numerical techniques are thought to be suggestive, we present them before presenting existence–uniqueness theories; sometimes when useful and not grossly misleading, we may even present them in lieu of existence–uniqueness theories. Also, we occasionally interrupt other presentations to give some theoretical background of basic computational procedures. However, any serious presentation of the theory of computation procedures is beyond the scope of this book. Nevertheless, we still have tried to present a text in which there is a natural integration of the topics of existence, uniqueness, approximation, and some analysis of computation procedures with applications.

PREFACE

Actually, our purpose has been to write a readable and teachable general text of modern mathematical science—one without substantial pretext to technical originality and yet one that is exciting and thorough enough to provide a basic background. The advantage of the modular approach is that a student may start where he finds his level, stop where his interests stop, and continue at his own rate, even piecemeal if he is so inclined. Courses can easily be organized from the text in the same manner. An instructor will find that he can easily addend or delete material without destroying the continuity of presentation. We believe, in fact, that most instructors want a text that will help them to organize their own course rather one that demands a specific approach. From our experience, we can recommend the following organizations of courses from this text, but we hope other instructors will find their own useful combinations of material and perhaps insert their own favorite topics:

- Mode 1: Part I, only—a one-quarter course for students of engineering and physics,
- Mode 2: Parts I and II—a one-semester, first-year graduate or senior-level course for students of mathematics, engineering, and physics,
- Mode 3: Parts I–III—a two-quarter, first-year graduate or senior-level course for students of mathematics, engineering, and physics,
- Mode 4: Parts I–IV—a two-semester, or three-quarter, first-year graduate or senior-level course for students of mathematics, engineering, and physics,
- Mode 5: Parts I-IV—a two-quarter (only) course at the third-year graduate level in mathematics (at this level, portions that review functional analysis, for example, can be skipped).

We have found it more pretentious than useful to present here a résumé of Lebesgue integration theory, but have, nevertheless, included a treatment of functional analysis that is fairly complete up to the point where it is required for our presentations. We believe the treatment is as brief and readable as one can find useful in the field. Except for our failure to present Lebesgue integration theory, which is really needed only in the last chapter (and even that can be bravely faced without it), we have kept our prerequisites down to just *some* introductory analysis beyond the level of the usual elementary calculus course and some elementary linear algebra. However, the instructor operating in Mode 1 may choose to dispense with even this requirement. I have taught these materials in Modes 1, 2, and 4. Professor Charles Bryan of the University of Montana, who as a doctoral student under my direction helped to write Chapter 15, has successfully used the materials from this text in Mode 5 (during 1969–1970). His assistance with the writing of this book has been most valuable, and Mode 5 is his idea. We must also acknowledge the influence of Professor Bernard Marcus of San Diego State College in the functional analysis chapter and Professor Robert Stevens of Montana University for a certain example we have used in Chapter 6; at the time when their contributions were made, both were engaged in doctoral studies under my direction.

The Mode 3 presentation does not involve Lebesgue integration beyond the level of some comments, mostly restricted to notes collected at the end of the text.

We have conceived Part I to be our "outline." Here we encourage the student to seek an understanding of the entire field of boundary-value problems by way of a more or less exhaustive study of the simplest linear homogeneous equations of the second order in two independent variables. This material must be completely understood before passing on to the study of the intensively analytic theorems of extensive generality that we find characterizes mature knowledge in this field. Our "outline" material is intended to provide breadth, not depth which, from our point of view, can only come in stages. Simply, successive parts of the book are designed to help in achieving successive levels.

Part II treats of existence and uniqueness by way of Picard iteration of the charteristic and Cauchy (initial value) problems for the wave equation in E^2 with its nonhomogeneous part depending, in a possibly nonlinear way, on the solution and its first partial derivatives. The Riemann method is developed, giving nonsingular integral representations for the linear case. This would seem to represent one of the admirable direct achievements of classical analysis, apparently motivated by Riemann's desire to understand flows of materials under large impact loadings and almost immediately applied by others to achieve an understanding of balanced transmission lines, foreshadowing the advent of clear long distance voice telephony. Transmission lines are studied in a separate chapter in Part II. Part II also treats the Cauchy–Kovalevsky theorem, which concerns analytic solutions of analytic equations corresponding to analytic data on a segment of an analytic initial value curve. The setting of Part II is thoroughly classical in conception throughout. It is quite important and unavoidably difficult in spots, even though it involves no advanced prerequisites.

In Part III we first sketch classical potential theory,* including the usual integral representations for the solution of the Dirichlet problem in terms of the Green function in n dimensions and a somewhat modern approach to variational principles for estimating quadratic functionals. The latter includes studies of such diverse topics as torsional rigidity and bounds for eigenvalues associated with some of the important boundary-value problems. It is then possible to move with ease to a study of the wave equation in higher dimensions, where the intriguing beauty of the Huyghens principle is emphasized and its inner workings exposed by using the Hadamard method of descent. Classical analysis eventually became heavily burdened with clever but extensive and delicate manipulations—presumably this overburden on analysis caused functional analysis to be conceived—and the latter portions of Part III unavoidably reflect this heavy manipulative style, but again it involves no advanced prerequisites.

Part IV presents a resumé of functional analysis, develops several a priori estimates for equations of elliptic–parabolic type (second order, n dimensions) from the

*This topic was once a semester or even year graduate course in mathematics.

divergence theorem, uses these to prove uniqueness of certain boundary-value problems, generates bounds therefrom for errors of approximation, and finally develops an **Abstract Existence Principle**, which is used to prove the existence of $\mathscr{L}^p$-weak solutions. This last assumes, but does not require, a rudimentary knowledge of Lebesgue integration theory. *Part IV is intended to provide in very specific terms a picture of the general techniques now being used in modern studies of partial differential equations.* At the end of Part IV, a discourse is undertaken concerning various modern senses of existence. Their physical relevance is reviewed, perhaps too briefly, but to the best of our ability. This is found to suggest a *possibility* that it is *perhaps* primarily the sense of uniqueness that we should think now to weaken—perhaps we should weaken it to time-asymptotic uniqueness with a quickly acquired (unique) steady state—retaining our classical (regular) sense of existence and at the same time insisting that all applied problems be treated as time dependent and not as stationary. If there is any technical, as opposed to expository, originality to be claimed for this text, it is the development of this thesis. We have tried, however, not to impose a private view onto a public body, simply asking that an awareness of such issues and an open mind concerning them be maintained. These, after all, are the issues raised in the last 20 years of progress in partial differential equations, and the effect of these 20 years has been so profound that the thinking in the field will never be the same again.

Perhaps the field of partial differential equations has suffered from too intense specialization among its adherents in the last several generations, but the danger now is too much generality taken on too fast by students without sufficient grounding in "real problems." We have tried here to introduce increased generality at a modest rate of increasing abstraction in stages that would seem to develop its justification in terms of problems that appear to be "real" at each stage.

Notes of general scientific and historical interest are collected at the end of the book (keyed to sections of various chapters) in order not to interrupt the flow of mathematical developments.

Contents

PART

I

An Outline

CHAPTER

1

The Theory of Characteristics, Classification, and the Wave Equation in E^2

1 D'ALEMBERT SOLUTION OF THE CAUCHY PROBLEM FOR THE HOMOGENEOUS WAVE EQUATION IN E^2

Let us consider under what conditions it is possible to determine a unique solution of the equation

$$u_{xx} - u_{yy} = 0 \tag{1.1.1}$$

satisfying the conditions

$$u(x,0) = f(x) \qquad \text{and} \qquad u_y(x,0) = g(x) \tag{1.1.2}$$

where $f\colon (a,b) \to R^1$ and $g\colon (a,b) \to R^1$. We understand that as part of this task we are to decide precisely what we wish to mean by saying a function u is a solution of (1.1.1), (1.1.2) and what properties given functions f and g must have so that such a solution exists and is unique. Toward this end we rewrite Eq. (1.1.1) in coordinates rotated through 45°,

$$\xi = \tfrac{1}{2}(x+y), \qquad \eta = \tfrac{1}{2}(x-y), \tag{1.1.3}$$

by the use of the chain rule on the function u. Let us remark here once and for all that there are really two functions involved, one a function of x and y, another a function of ξ and η formed as a composite of the first with (1.1.3), both of which have the same functional values at those points (x,y) and (ξ,η)

which are identified by (1.1.3). Because of this sameness of functional values arising by the formation of composite functions, we will use the same symbol u for both functions. Thus we may write $u(x, y)$ or $u(\xi, \eta)$ for functional values if the distinction of which function is used is not otherwise clear by the context, but the one symbol u will be used for both functions. Far from leading to confusion, as long as we agree to what is being done, this will help keep our bookkeeping straight as to which functional values are to be identified. This will be especially useful if we encounter long strings of changes of variables as one very often does in extensive application areas. As far as we know, all textbooks in partial differential equations are written using this convention, but in these times when the distinction between functions and their functional values is being greatly emphasized even in elementary training it would seem to need statement. In any case, it will be used throughout this text and not mentioned again unless clarification seems specifically demanded by the nature of the arguments presented.

From the chain rule we have

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = \tfrac{1}{2}(u_\xi + u_\eta) \\ u_y &= u_\xi \xi_y + u_\eta \eta_y = \tfrac{1}{2}(u_\xi - u_\eta) \\ u_{xx} &= \tfrac{1}{2}[(u_\xi + u_\eta)_\xi \xi_x + (u_\xi + u_\eta)_\eta \eta_x] \\ &= \tfrac{1}{4}(u_{\xi\xi} + u_{\eta\eta}) + \tfrac{1}{2}u_{\xi\eta} \\ u_{yy} &= \tfrac{1}{2}[(u_\xi - u_\eta)_\xi \xi_y + (u_\xi - u_\eta)_\eta \eta_y] \\ &= \tfrac{1}{4}(u_{\xi\xi} + u_{\eta\eta}) - \tfrac{1}{2}u_{\xi\eta} \end{aligned}$$

so that (1.1.1) becomes

$$u_{\xi\eta} = 0. \tag{1.1.4}$$

Here it has been assumed that $u_{\xi\eta}$ and $u_{\eta\xi}$ are continuous and are therefore equal; i.e., we are now restricted to seek a solution with this property.

We now seek the class of all solutions of (1.1.4). Equation (1.1.4) implies that u_ξ is a function of ξ alone. If this function is integrable, then we may write

$$u(\xi, \eta) = F(\xi) + G(\eta) \tag{1.1.5}$$

where G is an arbitrary function of η introduced by this last "integration" and F, being the primitive of u_ξ (a function of ξ alone), is also arbitrary.

We have thus shown that all solutions of (1.1.4) such that u_ξ is integrable are of the form (1.1.5). Now we must ask if all forms (1.1.5) are solutions of

(1.1.4). The question resolves to, what do we mean by a solution? Here we simply ask that all terms specifying quantities in (1.1.4) exist in some region R where this question is to be resolved and that (1.1.4) be satisfied in that region. But since we ask for an equality to be satisfied, we will also ask that all terms in the equation be continuous—here that $u_{\xi\eta}$ be continuous in the region of consideration. It is evident that the function u as given in (1.1.5) is a solution in this very concrete sense if $F, G \in C^1$ on sufficiently large open intervals of ξ and η.

Reverting back to the original coordinates, we have from (1.1.5)

$$u(x,y) = F(x+y) + G(x-y), \tag{1.1.6}$$

where the definitions of F and G have been altered to absorb the factor $\frac{1}{2}$ in the arguments. Here u is a solution of (1.1.1) in the sense that all terms exist and are continuous in a region of consideration if $F, G \in C^2$, and, moreover, if $F, G \in C^2(a,b)$, then one can see that $u \in C^2(T)$ where T is an isosceles triangle built on the base (a,b). Clarification of the latter will be undertaken in a moment; for now, we should note that the properties required of F and G in order that u in (1.1.5) be a solution in (ξ,η) coordinates are somewhat weaker than the properties required of them in order that (1.1.6) be a solution in (x,y) coordinates. This is a peculiar property of solutions of partial differential equations when considered in this very direct concrete sense, and it is one of the reasons (not the most cogent, however) why many modern workers prefer a more abstract sense of the existence of solutions. Such workers will be seen to lose much, however, in the way of useful physical interpretations of their results when they weaken the sense of existence. A balanced consideration of whether one should use the concrete sense (regular solutions, as we call them) or an abstract sense (e.g., $\mathscr{L}^p$-weak solutions) of the existence of solutions is a theme that will be threaded through this text but it has little relevance yet, and at first we are compelled to consider only the concrete sense of solutions. To some extent, where possible, it will be found that we prefer to weaken the sense of uniqueness rather than existence. Again that is far ahead of the story.

To find F and G in (1.1.6) so that (1.1.2) is satisfied we put

$$f(x) = F(x) + G(x) \tag{1.1.7}$$

and

$$g(x) = F'(x) - G'(x) \tag{1.1.8}$$

where $F(x+y)$ and $G(x-y)$ have been differentiated as composite functions of x and y and then y has been put equal to zero. Letting c be any real number,