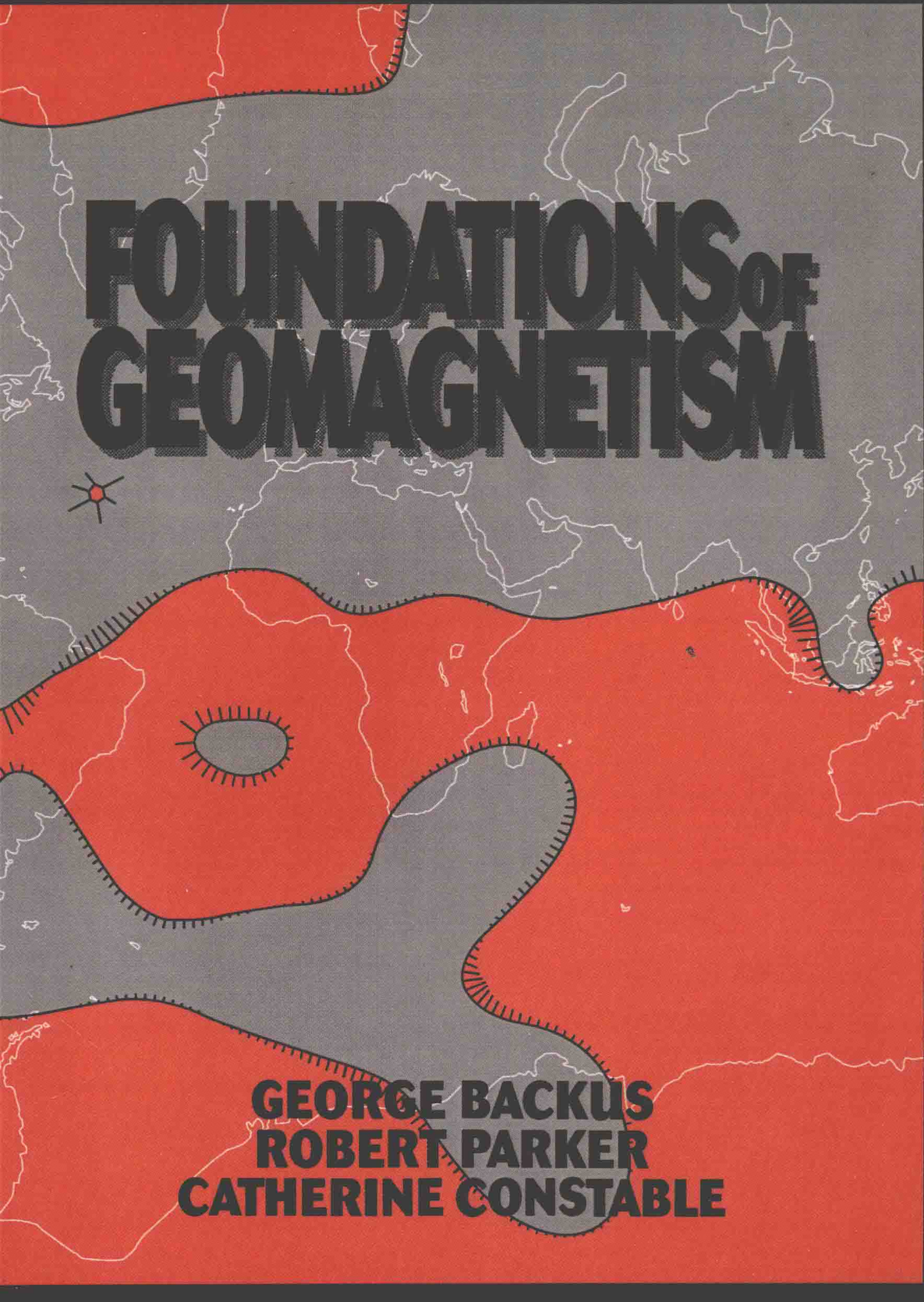

FOUNDATIONS OF GEOMAGNETISM

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521410069

First published 1996
This digitally printed first paperback version 2005

A catalogue record for this publication is available from the British Library

ISBN-13 978-0-521-41006-9 hardback
ISBN-10 0-521-41006-1 hardback

ISBN-13 978-0-521-01733-6 paperback
ISBN-10 0-521-01733-5 paperback

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PREFACE

The idea for this book was Philip Stark's: he suggested joining two of the authors, Bob Parker and Cathy Constable, in a project to "tidy up" George Backus's lecture notes and to make George a present of them in book form in time for his sixtieth birthday. Before this project got properly started, Philip left San Diego for Berkeley, and when the two of us still here began to look at the job seriously, we soon realized the timetable was not practical. Nonetheless, Cambridge University Press thought the idea was worth supporting and encouraged us to continue. The spirit of the enterprise was still that of offering George Backus a gift, rather than giving him more to do.

Over the years 1962 to 1994, George Backus has taught graduate level classes covering geomagnetism, mathematical techniques for geophysics, tensors in geophysics, and various other aspects of mathematical geophysics. Each time he taught, he would start from scratch, composing for the students hundreds of pages of closely spaced, hand written notes. He would devote himself entirely to the course each time, putting aside research and administration while it was being given. The level of the material was advanced, but the logical development, completeness, and consistency of notation were so compelling that generations of students came away enriched. Philip Stark's idea was to present something of this legacy to the world, but to spare George the tedium of editing and collating.

George's notes, on close inspection, comprise the solid mathematical skeleton of the material, but he himself provided the flesh of explanatory discourse and scientific context during his lectures. A major part of the editors' job has been to re-create the interpolatory text. Another part of our work was to decide what material to keep. Since we are both primarily geomagnetists, we felt more comfortable building around a geomagnetic theme rather than one of the other topics on which George has lectured at length, such as continuum mechanics or tensors. Even then, we could not keep everything; for example, the material on the Haar measure for averaging over a sphere from great circle paths could not be fitted neatly into the book. Only in a few places, most notably the second half of Chapter 3, have we felt the need to write new text to fill a gap.

The subject matter of *Foundations of Geomagnetism* is the mathematical and physical basis of the science of geomagnetism; graduate students in the earth sciences are its intended audience. George Backus has always been passionately concerned with the logical foundation of scientific argument and mathematical rigor in quantitative developments. Thus when he taught about the decomposition of a magnetic field into its poloidal and toroidal parts, he would never begin, "It can be shown that"; a proper account must start with the demonstration that a unique decomposition of this kind is always possible. Chapter 5 opens on this point. To build the foundations of geomagnetism, George calls upon some unusual mathematical tools, some of his own invention. The earth is nearly spherical, and geophysicists continually need to treat vectors and solve differential equations in spherical geometry. George shows how it is possible to maintain the elegance of a coordinate-free notation and at the same time to preserve the simplicity and familiarity of a Gibbs-like vector calculus for operators on a spherical surface. In this way he avoids the heavy baggage of the currently fashionable coordinate-free differential geometry à la Cartan. George's notation is intuitively right for the problem.

The final form of the book consists of seven chapters. The first is a brief overview of the phenomena that are of interest in geomagnetism. It includes a sketch of the history of the subject, followed by a description of the geomagnetic field and its variability in time and space. Chapter 2 concerns the classical theory of electromagnetism based on Maxwell's equations. We cover the physical and mathematical ideas of sources for the electromagnetic field and how the vacuum form of Maxwell's equations is adapted for polarizable media. We discuss the mathematical basis for the practice of neglecting small terms in an equation, and in particular we justify the neglect in geomagnetic work of the displacement current in Maxwell's equations. The discussion of sources introduces the concept of the separation of the magnetic field into parts of internal and external origin.

Chapter 3 is devoted to spherical harmonics. The aim is to develop from first principles all the standard results. The perspective and methods used are not the traditional ones, however. In particular, the spherical harmonics of degree ℓ are exhibited as homogeneous harmonic polynomials in x, y, and z, which are treated as members of a finite-dimensional vector space. We introduce an inner product on the space and then consider an orthogonal basis for it. Various linear operators mapping this space onto itself are used to explore the symmetries of the system, and thereby to discover its properties, such as the Addition Theorem, and the existence of a self-reproducing kernel. Up to this point, the results have been independent of the particular coordinate system. Once a special axis system is chosen, we can develop explicit expressions for the traditional set of orthonormal functions. We investigate the asymptotic properties of the functions, derive recurrence relations among them, and describe a scheme for their practical computation.

Chapter 4 gives the application of spherical harmonics in the description of the main geomagnetic field. We study the question of the uniqueness of the coefficients in a spherical harmonic expansion containing internal and external parts. Other topics include the geomagnetic power spectrum, downward continuation to the core, and how little information about the sources is contained in the Gauss coefficients.

The subject of Chapter 5 is the Mie representation, which is the expression of a solenoidal field as a sum of poloidal and toroidal parts. Again the theory is developed from first principles, beginning with the Helmholtz representation for tangent vector fields on a spherical surface, a useful representation in its own right. The Mie representation is then applied to a variety of geomagnetic problems, including the generalization of Gauss' separation to regions containing sources, outward diffusion of the toroidal magnetic field of the core, geomagnetic sounding, and the free decay of magnetic fields in a stationary core.

The material of the preceding chapters is brought to bear in Chapter 6, where we consider the physical processes taking place in the core of the earth. After a quick look at a simplified dynamo model with only two degrees of freedom, we derive the full system of partial differential equations governing the interaction of a moving conducting fluid with an embedded magnetic field. To obtain the version of Ohm's law needed in a moving conductor, we must appeal to relativistic physics. The two descriptions of a continuum, Lagrangian and Eulerian, are discussed and their relationship exposed.

Two limiting problems are solved exactly: zero velocity and infinite conductivity. The latter is shown to be a useful approximation in the earth for time scales less than 100 years, and therefore of considerable interest in the interpretation of the secular variation. We derive the "frozen-flux" condition of Roberts and Scott, which, if valid, permits us to deduce from the magnetic field what the fluid velocity is on the lines where the radial field vanishes. The chapter closes with a sample of dynamo theory, a mixture of brief qualitative summaries of some important results together with a few topics laid out in mathematical detail, including Cowling's antidynamo theorem and a glimpse at mean field dynamos, currently so popular.

The Mie representation and, to a lesser extent, spherical harmonics are dependent on vector calculus on the surface of a sphere. Chapter 7 is a compendium of mathematical theorems and results needed elsewhere in the book. The chapter provides a general treatment of linear operators that act on scalar and vector fields. The general theory is specialized to the case of a spherical surface, and the properties of those most useful operators, surface gradient and surface curl, are then developed in some detail. Corresponding results for more general surfaces are touched upon. Other topics in Chapter 7 include surface forms of the integral theorems of Stokes and Gauss, and inclusion of jump discontinuities in those theorems and how this affects the Helmholtz Theorem.

The two junior authors have both learned an enormous amount by going over this material in detail. We can only hope some of the craftsmanship and the intellectual discipline demonstrated by this work has rubbed off on us. We are grateful to Philip Stark for suggesting the project. We would also like to express our thanks to Elaine Blackmore, who translated the hand written notes into TEX with great speed and accuracy, all the more amazing given that this was her first experience with TEX. We wish to express our gratitude to the Director of Scripps Institution of Oceanography for the financial help he provided as we were getting started. Cambridge University Press and its editors deserve our gratitude for their patience and understanding, as well as their enthusiasm for the idea. Once again, we want to thank the senior author, George Backus, for his example as a great scientist and a warm human being.

R. Parker
C. Constable

CONTENTS

1

THE MAIN FIELD

1.1 A Whirlwind Tour

In this chapter we go on a whirlwind tour of the subject matter of geomagnetism. Far from being a comprehensive survey, this is an outline of the observations and phenomena that geomagnetism aims to understand. It is expected that every serious student of our science will be familiar with everything in this chapter, but those from outside the earth sciences may find the following summary helpful. The following references will help fill in the gaps.

For an accessible yet scholarly summary of the history of geomagnetism, one can find nothing that improves upon the final chapter in Volume II of *Geomagnetism* by Chapman and Bartels (1962). The short essay by Malin in the more recent *Geomagnetism*, edited by Jacobs (1987), is another useful source. Appendix B of the paper by Malin and Bullard (1981) gives a series of thumbnail sketches of the important players in geomagnetism in Great Britain from 1570 to 1900. A fascinating history of the discovery of the reversing nature of the main field can be found in *The Road to Jaramillo* (Glen, 1982).

A review of the spatial variation of the main field and the crustal magnetic anomalies can be found in Jacobs's *Geomagnetism* (1987). The practical techniques of magnetic data interpretation, particularly for fields originating in the crust, are thoroughly covered by Blakely (1995). A detailed account of the phenomena of the magnetosphere and the ionosphere is presented in *Physics of Geomagnetic Phenomena*, edited by Matsushita and Campbell (1967); a more modern but less detailed treatment is given by Parkinson (1983). These works are also good places to read about the externally caused time variations, but for detail of a purely descriptive kind, Volume I of Chapman and

Bartels (1962) remains the classic source. An excellent treatment of the longer-period time variations is to be found in the monograph by Merrill and McElhinny (1983), which also covers our other topics, though with somewhat less authority.

1.2 History

The existence of magnetic forces, through the tendency of lodestones to attract iron, has been known for perhaps 4000 years, having first been noted in China. Lodestones, which are naturally magnetic pieces of magnetite, are mentioned by Homer (ca 800 BC) and by later Greek and Roman writers such as Pliny the Elder (24–79 AD). The use of the magnetic field for navigational purposes cannot be unequivocally identified until 1088 in China and nearly 100 years later in Europe. Believing in the perfection of celestial phenomena, the early navigators assumed a compass would point exactly to geographic north. But of course this is mistaken, and the discrepancy, called variation in the original accounts but now referred to as *declination*, was generally recognized by the middle of the fifteenth century; its discovery in Europe is sometimes erroneously credited to Christopher Columbus. In 1581, Robert Norman (dates uncertain), a London instrument maker, reported the fact that the true direction of the field was not horizontal and that a compass needle, carefully balanced before being magnetized, would point downward, or dip. Thus magnetic *inclination*, as it is now called, was discovered.

It is time to introduce the first mathematical terms, the so-called geomagnetic elements that describe the magnetic field vector at a given point on the earth's surface. The following are the traditional names still in common use in the study of geomagnetism. We consider a local Cartesian coordinate system with x pointing to geographic north, y to the east, and z vertically downward. The magnetic elements X, Y, Z are the components of the magnetic field vector $\mathbf{B}$ in this frame. (We will define exactly what a magnetic field vector is in the next chapter.) Then the declination D is obtained from

$$\tan D = Y/X. \tag{1.2.1}$$

The total intensity, variously referred to as T or F, is obviously

$$F = \sqrt{X^2 + Y^2 + Z^2}. \tag{1.2.2}$$

The horizontal intensity H is just

$$H = \sqrt{X^2 + Y^2}. \tag{1.2.3}$$

Inclination, I, satisfies the equation

$$\tan I \doteq Z/H. \tag{1.2.4}$$

Worldwide exploration by European navigators who measured declination and sometimes even inclination of the magnetic field enabled William Gilbert (1540–1603), chief physician to Queen Elizabeth I, to assemble a global picture of the field directions and to deduce that "The earth is a great magnet." He showed that a spherical lodestone, which he called a *terrella*, was surrounded by a magnetic field whose directional properties closely resembled those of the earth's field. His monumental book, *De Magnete* published in 1600, is widely regarded as the first scientific text, being entirely free of the appeals to heavenly causes and magic that were the common currency of explanation of the day.

Although it was probably suspected when Gilbert wrote his treatise, the fact that the earth's field changes in time was explicitly denied in *De Magnete.* In 1624, Edmund Gunter (1581–1626), professor of astronomy at Gresham College, had collected observations that pointed strongly to temporal variations in declination. His successor, Henry Gellibrand (1597–1636), completed the study after Gunter's death and published the discovery of secular variation in 1635. By 1680, Edmund Halley (1656–1742) produced an amazingly prescient model for the variation in terms of dipoles moving generally westward, deep within the earth, making a circuit every 700 years. The westward drift of small-scale features of the geomagnetic field is an important clue about fluid motions of the core; today the period is estimated to be about 2000 years.

An almost completely modern description of the main geomagnetic field was obtained by Karl Friedrich Gauss (1777–1855), the great German mathematician and geomagnetist. He completed the notion of the magnetic field as a vector by defining and determining its strength. Gauss invented spherical harmonics for the description of the field (a subject that will occupy us later on) and deduced that the origin of the main field is within the solid earth, not outside it, thus confirming Gilbert's early speculation. We will repeat an updated form of Gauss' argument in Chapter 4. Gauss was responsible for setting up a worldwide system of magnetic observatories, some of which have been running continuously up to the present day. Of course, the cgs unit of magnetic induction is named for Gauss, although we adhere to the Système Internationale (SI) convention in which 1 tesla = 10^4 gauss. Unlike the gauss, the tesla, named for the Polish-American electrical engineer Nicola Tesla (1856–1943), is an inconvenient size for geomagnetic use; generally the field intensities are referred to in nanotesla (1 nT = 10^{-9} tesla) or microtesla (1 μT = 10^{-6} tesla). It was Gauss who first observed that strength of the main field at the surface varies from a maximum of about 60 μT (or 0.6 gauss) at the poles to about 30 μT at the equator. His theoretical results predicted that the intensity drops off approximately as the inverse cube of the distance from the earth's center.

The major phenomenological discovery of the twentieth century about the main geomagnetic field is the fact that the prominent dipole has reversed polarity many hundreds of times during the earth's geological history. The study of the magnetization of rocks in France and Italy led Bernard Brunhes (1867–1910) to conclude in 1906 that the ancient magnetic field had the opposite direction from today's. In 1926 Paul Mercanton (1876–1963) came to the same conclusion regarding rocks from Spitzbergen and Australia; Motonori Matuyama (1884–1958) also deduced this from specimens taken in Japan and Siberia. Yet for over half a century this evidence was not generally regarded as conclusive for a variety of reasons. Most important, it was not at all clear that rocks could retain magnetism gained millions of years ago. Today much more about the physics of rock magnetism is known and there is no real doubt that most rocks containing magnetic minerals, both igneous and sedimentary, can record and preserve almost indefinitely information about the field at the time when they were formed. Stacey and Banerjee (1974) and O'Reilly (1984) provide comprehensive treatments of the physics of rock magnetism. By 1963, Cox, Doell, and Dalrymple had compelling evidence of reversals based on radiometric ages; their measurements demonstrated conclusively that the field was reversed all over the globe during a number of well-defined epochs stretching back 4 million years; see Cox, et al. (1964). In 1963 Vine and Matthews recognized the regular magnetic anomaly stripes they had observed in the Indian Ocean as the record of the reversing field in the seafloor rocks. Aside from the revolutionary implications for geology, this great insight made it possible to establish