Graduate Texts in Mathematics

William S. Massey

A Basic Course in Algebraic Topology

代数拓扑基础教程

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Preface

This book is intended to serve as a textbook for a course in algebraic topology at the beginning graduate level. The main topics covered are the classification of compact 2-manifolds, the fundamental group, covering spaces, singular homology theory, and singular cohomology theory (including cup products and the duality theorems of Poincaré and Alexander). It consists of material from the first five chapters of the author's earlier book *Algebraic Topology: An Introduction* (GTM 56) together with almost all of his book *Singular Homology Theory* (GTM 70). This material from the two earlier books has been revised, corrected, and brought up to date. There is enough here for a full-year course.

The author has tried to give a straightforward treatment of the subject matter, stripped of all unnecessary definitions, terminology, and technical machinery. He has also tried, wherever feasible, to emphasize the geometric motivation behind the various concepts. Several applications of the methods of algebraic topology to concrete geometrical–topological problems are given (e.g., Brouwer fixed point theorem, Brouwer–Jordan separation theorem, Invariance of Domain, Borsuk–Ulam theorem).

In the minds of some people, algebraic topology is a subject which is "esoteric, specialized, and disjoint from the overall sweep of mathematical thought." It is the author's fervent hope that the emphasis on the geometric motivation for the various concepts, together with the examples of the applications of the subject will help to dispel this point of view.

The concepts and methods which are introduced are developed to the point where they can actually be used to solve problems. For example, after defining the fundamental group, the Seifert–Van Kampen theorem is introduced and explained. This is the principal tool available for actually determining the structure of the fundamental group of various spaces. Another such example is the cup product. Not only is the cup product defined and its principal properties explained; cup products are actually determined in real, complex, and quaternionic projective spaces, and these computations are then applied to prove certain theorems.

In any exposition of a subject such as algebraic topology, the author has to make choices at various stages. One such choice concerns the class of spaces which will be emphasized. We have preferred to emphasize CW-complexes rather than simplicial complexes. Another choice occurs in the actual definition of singular homology groups: Should one use singular simplices or singular cubes? From a strictly logical point of view it does not matter because the resulting homology and cohomology theories are isomorphic in all respects. From a pedagogical point of view, it does make a difference, however. In developing some of the basic properties of homology theory, such as the homotopy property and the excision property, it is easier and quicker to use the cubical theory. For that reason, we have chosen to use the cubical theory. Of course, it is more traditional to use the simplicial theory; the author hopes that possible prospective users of this book will not reject it because of their respect for tradition alone.

The prospective user of this book can gain some idea of the material contained in each chapter by glancing at the Contents. We are now going to offer additional comments on some of the chapters.

In Chapter I, the classification theorem for compact 2-manifolds is discussed and explained. The proof of the theorem is by rather standard "cut and paste" methods. While this chapter may not be *logically* necessary for the rest of the book, it should not be skipped entirely because 2-manifolds provide a rich source of examples throughout the book.

The general idea of a "universal mapping problem" is a unifying theme in Chapters III and IV. In Chapter III this idea is used in the definition of free groups and free products of groups. Students who are familiar with these concepts can skip this chapter. In Chapter IV the Seifert–Van Kampen theorem on the fundamental group of the union of two spaces is stated in terms of the solution to a certain universal mapping problem. Various special cases and examples are discussed in some detail.

The discussion of homology theory starts in Chapter VI, which contains a summary of some of the basic properties of homology groups, and a survey of some of the problems which originally motivated the development of homology theory. While this chapter is not a prerequisite for the following chapters from a strictly logical point of view, it should be extremely helpful to students who are new to the subject.

Chapters VII, VIII, and IX are concerned solely with singular homology with integer coefficients, perhaps the most basic aspect of the subject. Chapter VIII gives various examples and applications of homology theory, including a proof of the general Jordan–Brouwer separation theorem, and Brouwer's theorem on "Invariance of Domain." Chapter IX explains a systematic method of computing the integral homology groups of a regular CW-complex.

In Chapter X we introduce homology with arbitrary coefficient groups. This generalization is carried out by a systematic use of tensor products. Tensor products also play a significant role in Chapter XI, which is concerned with the homology groups of a product space, i.e., the Künneth theorem and the Eilenberg–Zilber theorem.

Cohomology groups make their first appearance in Chapter XII. Much of this chapter of necessity depends on a systematic use of the Hom functor. However, there is also a discussion of the geometric interpretation of cochains and cocycles, a subject which is usually neglected. Chapter XIII contains a systematic discussion of the various products: cup product, cap product, cross product, etc. The cap product is used in Chapter XIV for the statement and proof of the Poincaré duality theorem for manifolds. This chapter also contains the famous Alexander duality theorem and the Lefschetz–Poincaré duality theorem for manifolds with boundary. In Chapter XV we determine cup products in real, complex, and quaternionic projective spaces. These products are then used to prove the classical Borsuk–Ulam theorem, and to give a discussion of the Hopf Invariant of a map of a $(2n - 1)$-sphere onto an n-sphere.

The book ends with two appendices. Appendix A is devoted to a proof of the famous theorem of DeRham, and Appendix B summarizes various basic facts about permutation groups which are needed in Chapter V on covering spaces.

At the end of many chapters there are notes which give further comments on the subject matter, hints of more recent developments, or a brief history of some of the ideas.

As mentioned above, there is enough material in this book for a full-year course in algebraic topology. For a shorter course, Chapters I–VIII would give a good introduction to many of the basic ideas. Another possibility for a shorter course would be to use Chapter I, skip Chapters II through V, and then take as many chapters after Chapter V as time permits. The author has tried both of these shorter programs several times with good results.

Prerequisites

As in any book on algebraic topology, a knowledge of the basic facts of point set topology is necessary. The reader should feel comfortable with such notions as continuity, compactness, connectedness, homeomorphism, product space, etc. From time to time we have found it necessary to make use of the quotient space or identification space topology; this subject is discussed in the more comprehensive textbooks on point set topology.

The amount of algebra the reader will need depends on how far along he is in the book; in general, the farther he goes, the more algebraic knowledge will be necessary. For Chapters II through V, only a basic, general knowledge of group theory is necessary. Here the reader must understand such terms as group, subgroup, normal subgroup, homomorphism, quotient group, coset, abelian group, and cyclic group. Moreover, it is hoped that he has seen enough examples and worked enough exercises to have some feeling for the true significance of these concepts. Most of the additional topics needed in group theory are developed in Chapter III and in Appendix B. Most of the groups which occur in these chapters are written multiplicatively.

From Chapter VI to the end of the book, most of the groups which occur are abelian and are written additively. It would be desirable if the reader were familiar with the structure theorem for finitely generated abelian groups (see Theorem III.3.6). Starting in Chapter X, the tensor product of abelian groups is used; and from Chapter XII on the Hom functor is used. Also needed in a few places are the first derived functors of tensor product and Hom (the functors Tor and Ext). These functors are described in detail in books on homological algebra and various other texts. At the appropriate places we give complete references and a summary of their basic properties. In these later chapters we also use some of the language of category theory for the sake of convenience; however, no results or theorems of category theory are used. In order to read Appendix A the reader must be familiar with differential forms and differentiable manifolds.

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William S. Massey
New Haven, Connecticut
March, 1988

Notation and Terminology

The standard language and notation of set theory is used throughout. Some more special notations that are used in this book are the following:

$\mathbf{Z}$ = ring of integers,
$\mathbf{Q}$ = field of all rational numbers,
$\mathbf{R}$ = field of all real numbers,
$\mathbf{C}$ = field of all complex numbers,
$\mathbf{R}^n$ = set of all n-tuples $(x_1, \ldots, x_n)$ of real numbers,
$\mathbf{C}^n$ = set of all n-tuples of complex numbers.

If $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, then the *norm* or *absolute value* of x is

$$|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

With this notation, we define the following standard subsets of $\mathbf{R}^n$ for any $n > 0$:

$$E^n = \{x \in \mathbf{R}^n \mid |x| \leqq 1\},$$

$$U^n = \{x \in \mathbf{R}^n \mid |x| < 1\},$$

$$S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}.$$

These spaces are called the *closed n-dimensional disc* or *ball*, the *open n-dimensional disc* or *ball*, and the *$(n - 1)$-dimensional sphere*, respectively. Each is topologized as a subset of Euclidean n-space, $\mathbf{R}^n$. The symbols RP^n, CP^n, and QP^n are introduced in Chapter IX to denote n-dimensional real, complex, and quaternionic projective space, respectively.

A homomorphism from one group to another is called an *epimorphism* if it is onto, a *monomorphism* if it is one-to-one (i.e., the kernel consists of a single element) and an *isomorphism* if it is both one-to-one and onto. If $h : A \to B$ is a homomorphism of abelian groups, the *cokernel* of h is the quotient group $B/h(A)$. A sequence of groups and homomorphisms such as

$$\cdots \longrightarrow A_{n-1} \xrightarrow{h_{n-1}} A_n \xrightarrow{h_n} A_{n+1} \longrightarrow \cdots$$

is called *exact* if the kernel of each homomorphism is precisely the same as the image of the preceding homomorphism. Such exact sequences play a big role from Chapter VII on.

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