George K. Francis

A Topological Picturebook

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A Topological Picturebook

With 87 Illustrations



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To Bettina

PREFACE

Die Kräftigung des räumlichen Vorstellungsvermögens und der räumlichen Gestaltungskraft gehört unbestritten zu den wichtigsten Zielen eines jeden geometrischen Unterrichts.
Artur Schoenflies, 1908

Surely, among the most important goals of every geometrical instruction is the strengthening of the faculty for spatial imaging and the power for spatial modelling.

My book is about how to draw mathematical pictures. Of course, it begs the question of whether one should draw pictures in mathematics in the first place. Some disciplines require illustrations. Imagine Gray's *Anatomy* without pictures. Some get along better without them. Imagine trying to illustrate Russell and Whitehead's *Principia*.

The decade of this book's gestation began with a sabbatical semester I spent in our library with the master geometers of the previous century. In the pages of the *Enzyclopaedie der Mathematischen Wissenschaften* and the collected works of Felix Klein I began to see, through hundred-year old eyes, what I had learned as a mathematics student in the middle of this century.

Theirs was a wonderfully straightforward way of looking at rather complicated things, notably Riemann surfaces and geometrical constructions over the complex numbers. They drew pictures, built models and wrote manuals on how to do this. And so they also captured a vivid record of the mathematics of their day. I resolved to try to do the same for the mathematics of my contemporaries.

In my own work I had always relied heavily on sketches and diagrams. I learned really to appreciate this experimental way of doing mathematics during my Ann Arbor days. There Chuck Titus guided me skillfully through my thesis by drawing hundreds of pictures. I also adopted his historical approach towards geometrical analysis, which leads to a harmony between logically abstract and visually concrete mathematics. Encouraged by the

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leaders in my field, Morris Marx in particular, I entertained my lecture and seminar audiences with elaborate blackboard drawings and pictorial "souvenir" handouts. Eventually, I began to work on pictures of things other people wanted to have portrayed. I had the pleasure of illustrating an article by Berstein and Edmonds [1979], a Springer monograph by Bill Abikoff [1980] and his essay in the *Monthly* [1981], two papers by Bill Thurston [1982,1986] and his as-yet-unpublished textbook, *The Geometry and Topology of Three-Manifolds*. And, above all, I now actively badgered my colleagues to tell me their secret visions with the promise of showing them a simple way of sketching them on demand.

When I first decided to formalize the procedures I had developed for drawing useful mathematical pictures, I planned on calling the result "A Graphical Calculus." I still believe that there are some rules, based on differential geometry, which can be distilled into practical routines for "calculating" how to draw a picture. In the meantime, however, I have come to favor a less didactic, more advisory purpose for setting down any rules. You should regard this book as a description of my own graphical calculus and as an invitation for you to modify it and improve upon it to suit your own purposes.

The pictures in this book are certainly meant to be copied or reproduced, peferably by hand. They also can be traced onto a transparency. Most pen drawings "work" equally well without the shading I have applied to them for effect. Each picture has a "line pattern" from which it was constructed. In many instances I included this pattern explicitly. In others you may rediscover it by analysing the finished picture. The line pattern is usually simple enough to commit to memory or to your lecture notes. The pictures have been tested out at the blackboard to make sure that this mode of presentation is also feasible. Even the more elaborate ones, using colored chalk, can be reproduced in half an hour. Yet, the pedagogical advantage of composing a picture at the board, erasing the hidden lines, highlighting the curvature and using colored chalk to label details is compromised by the impermanence of this time-consuming medium. The only efficient way to use it for talks is to make photographic slides of the drawings. Therefore the book concentrates on how to make india-ink line drawings that are robust enough to survive several xerographic generations and simple enough to reproduce by hand.

A word about the format I chose and how this topological picture-storybook got its name. I still have the *Märchen-Bilderbuch* my grandfather gave me for my third birthday. He had bound it himself. My oldest possession, this beautifully illustrated volume of Grimms' tales [1942], survived wartime bombs and postwar turmoil. I have always loved this literary genre. So when my patient and generous Springer editor, Walter Kaufmann-Bühler, suggested I add more text by way of explanation, it seemed natural to tell topological stories to go with the pictures.

This way each chapter became a "picture story" about something in a

particular area of mathematics. Each owes its telling to expert friends who one way or another told me about their territory. Serendipity selected the subjects that came my way. But responsibility for the text, whether clear or not, as well as for the drawings, whether pleasing or not, is strictly my own.

Chapter 1 establishes the relationship between Whitney and Thom's theory of general position, of stable singularities of low-dimensional mappings, and descriptive topology. The drawings were originally prepared for Robin McLeod's Workshop on Frontiers of Applied Geometry at Las Cruces, New Mexico in 1980. I have given this lecture many times since and directed it especially to those who draw pictures to go along with their own lectures.

The techniques in this and the next chapter were originally collected for my efforts in using pictures to explain what Catastrophe Theory is all about. By such visual means I was able to "explain" Thom and Zeeman's intriguing invention also to non-topologists, in the sense that I was able to arouse and satisfy their curiosity. My courses for high-school math teachers in Peoria and biology students at the University and my seminars for physics and physiology colleagues all relied on the *lingua franca* of accurately drawn pictures. I found these pictures in the literature and learned how to reproduce them from the drawing of Jim Callahan, Tim Poston and Les Lander in Callahan [1974,1977], Poston-Stewart [1978], and Bröcker-Lander [1975] respectively.

Chapter 2 is the *media and methods* section of the project, a form I borrowed from the laboratory sciences. Topologists usually draw pictures to help formulate their definitions and proofs. On the way to publication, these visual props are all too often left behind. Here then is a fair sample of the various ways a topologist may produce pictures suitable for publication. This chapter is more of a collection of topological "anecdotes" than a coherent picture story. Most of them were originally prepared for Bill Abi' off's long lived and much-appreciated Riemann Seminar, where we studied the work of Thurston *inter alia*. The last five figures in this chapter were "commissioned" executions of Thurston's sketches.

I am often asked whether my pictures are drawn by a computer. Not yet! My efforts first to learn and then to enlist this difficult medium into the service of topology began as long ago as the picturebook project. How I used computer output for my drawings is explained herein. I wish to thank my tutors, Judy and Bruce Sherwood, who initiated me into the electronic mysteries of PLATO. Only recently has it been possible for me to use a computer with anywhere near the ease and facility of a pen. I have included a few color photographs of computer-drawn surfaces designed in a manner outlined in this book. They were made by Donna Cox and Ray Idaszak on machines provided by the Electronic Imaging Lab and the National Center for Supercomputing Applications at the University of Illinois. My postscript, of a somewhat technical nature, tells the story of these pictures.

Chapter 3 is a review of the principles of perspective drawing from the viewpoint of projective geometry. Its terse and didactic style is borrowed

from descriptive geometry. It centers on three elementary problems a draftsman faces when drawing in freehand perspective. I learned about these and their various solutions from Norm MacFarland through the University of Illinois Program for Faculty Study of a Second Discipline. The semester I was a guest in Ed Zagorski's Department of Industrial Design introduced me to an artistic discipline with remarkably strong affinity to descriptive topology. Invaluable drafting lessons from Vivian Faulkner-King led to my redesigning some and redrawing most of the older pictures.

For completeness I added a section on non-perspective, so called "engineering" drawing, which — unfortunately — is still used by many publishers of mathematical textbooks. This style, with its illogical distortions, was originally developed to convey metric information from designer to machinist. Computer aided design does this much better nowadays, and computer graphics makes perspective views trivially accessible to all. With some reluctance I also included a section on shading. Despite many hours of careful inspection of Escher, Dürer, and Dali's works, I don't think I have mastered the technique of light and shade well enough to teach it.

In this chapter more than in the others I have tried to speak the language of the artist and designer. While the main purpose of this book is to persuade fellow topologists to draw the pictures that illustrate their work, I hope it will also inspire artists to study topology and apply their skill to this wonderfully visual discipline.

Chapter 4 is a short story about the optical illusion on which several of Maurits Escher's best known prints are based. Here it serves as an introduction to the differential geometry of spaces of constant curvature. Originally I distributed the drawings as a geometric curiosity. At times I thought the impossible tribar had a natural home in a lens space. For the true story, I am indebted to many colleagues who spotted such errors and suggested improvements, notably Wolfgang Haken, Allan Hatcher, Dave Fried and John Stillwell.

Chapter 5 is algebraic geometry in a most elementary and practical sense: just what should a picture of an algebraic operation, such as squaring or multiplying, really look like? Building on common knowledge of ninth-grade analytic geometry, which gives answers over the field of real numbers, I draw the shadows of four-dimensional shapes that give answers over the complex number field. My debt to the classic computer graphics films by Tom Banchoff and Charles Strauss [1977] is clear. This chapter was particularly influenced by their treatment of the complex function graphs and the Veronese Surface. However, the pedagogy that underlies the entire book, and which I bring out specially here, comes from Bernard Morin of Strasbourg. Pictures without formulas mislead, formulas without pictures confuse. I don't know if Bernard would say it this way, but it is how I have understood his work.

The chapter concludes with applications of the graphics and of the mathematics. In the former I add some pictures to the rarely referenced, highly visual part of Whitney's famous treatise [1941], where he gives a

concrete meaning to his theory of characteristic classes. The latter refers to the splendid thesis by the Morin student, François Apery [1984]. It constitutes a gateway between traditional hand-drawn graphics and efficient computer graphics, a subject to which I return in the postscript.

Chapter 6 is, in a sense, the first picture story. It relates once more, and with a different emphasis, the details of a famous visualization problem in differential topology. When I tried to explain the sphere eversion to my freshman topology seminar in 1977 I already had the *Scientific American* cover article by Tony Phillips [1966] and the NSF-sponsored computer graphics film by Nelson Max [1977] at my disposal. Still, it took much effort and many blackboard sessions to show my students what they had, in fact, already seen. In time it became clear that the few ink drawings that fit on a single page could suffice to demonstrate an eversion; see pages 124 and 113. The design of these pictures was based on a well-established body of theorems on the classification of excellent mappings, Francis-Troyer [1977,1982]. Thus the demonstration of the tobacco pouch eversion could be made as rigorous as one's taste dictated. In the end, however, it was Bernard himself who provided the analytic parametrization for this remarkable deformation of a surface; see Morin [1978]. I hope that you too will recognize the intimate relation of topology, art and analytic geometry, at least in this singular case.

Chapter 7 began as an orphan. In the course of studying Hatcher and Thurston's proposed scheme for presentating the mapping class group of an arbitrary surface I had drawn some pictures for our Riemann Seminar. They illustrated details in the notes by André Marin [1977], which at that time were circulating in the mathematical *samizdat*. Already in the first manuscript circulated by the authors, and decidedly in their final version [1980], a revision of their argument made my pictures largely superfluous. I decided I would instead apply my efforts to illustrating the topology behind the well-known presentation of the mapping class group of the double torus. Combinatorial group theory is a field the furthest from my talents that I ever dared approach. That I succeeded in my quest is to the credit of my mentors, Bill Abikoff, Joan Birman, John Ratcliffe, Paul Schupp, and especially Wilhelm Magnus, who provided endless hours of patient explanation to go with a generous gift of his collected works. I hope that the pictures in this chapter are only the beginning of an endeavor, joined by others, to enrich the visual side of this abstract but beautiful mathematical discipline.

Chapter 8 is about a fibered knot and so belongs to geometric topology. Since the knot is the star of Thurston's book [1977,1982], my story belongs to the larger domain of low-dimensional geometry and topology. At any rate, here it is again together with drawings and side trips that I left out of its original version, published in the *Monthly* [1983]. That article was a trial balloon for an expository style suitable for the picturebook. The excellent suggestions of its editor, Paul Halmos, its four referees (two known to me,

two not), and the kind comments of its readers eventually fused into the genre I call a *topological picture story.*

I don't think the picturebook manuscript would ever have been finished without the helpful criticism of friendly readers and the persistence of my publisher. It would have remained an embarrassment of obscurity without the veritable flood of red ink, for which I am eternally grateful to Ian Stewart. The project has also older debts. Early and awkward versions of most ideas herein were foisted on and loyally dissected by fellow members of the Geometric Potpourri Seminar: Ralph and Stephanie Alexander, Felix Albrecht, Dick Bishop and Dave Berg. Hermann Rusius of Enschede, Holland provided a rare sort of encouragement which deserves special mention. Cloth fabrics merchant by profession, self-taught geometer and builder of mathematical models, in the winter of 1979 he arrived in Urbana in a blizzard and left again with the next. In between he photographed the dusty plaster models of algebraic surfaces made by Verlag Schilling of Leipzig in the last century that now lie entombed in the glass cases of Altgeld Hall. We spent many evenings together talking shop. Later, when it came time to write about my pictures and I was at a loss to think of my proper audience I secretly imagined Mr. Rusius and tried to explain it to him.

And of course, no project of this absorbing a nature could have been sustained over such a lenghty span of time without long-suffering forebearance and cheerful support from my wife and children.

At critical moments in the production of this book I received expert help from many old and new friends in the book printing profession. Special thanks go to editors Rob Torop and Barbara Tompkins of Springer-Verlag, to Carole Appel, senior editor at the University of Illinois Press, and to Larry Lutz, Carl Kibler and their colleagues at the University of Illinois Print Shop, where the glow of electronic typography reflects off venerable Monotype machines, and the smell of photographic chemicals mixes with the nostalgic odor of hot lead.

Urbana, Illinois
September 1986

George K. Francis

CONTENTS

Reference to a detail of a figure in the current chapter is by number, row and column. If the figure is located in another chapter then its number preceeds the cross reference. Thus the Möbius band in the lower right corner of Figure 9 in Chapter 6 is called 6:9(34). Chapter subheadings which are also figure captions are preceded by the figure number and followed by their page number.

COLOR PLATES (following page 16)

I Blackboard Dunce Cap
II Diapered Trefoil Knot
III Cross Cap and Limaçon
IV Etruscan Venus

Chapter 1 DESCRIPTIVE TOPOLOGY 1

1 Saddle in a Box 2
2 Saddle in a Drum 4
3 Whitney Umbrella 7
4 Cayley Cusp 10
5 Pinch Point/Branch Point 12

Chapter 2 METHODS AND MEDIA 14

1 Cubic Metaphor 15
2 Basic Surface Patterns 17
3 Triprong 18
4 Rounding a Polyhedral Saddle 19
5 Dunce Hat 21
6 Duns Egg 23
7 Möbius Bands 25
Ink and Paper 24
Chalk and Blackboard 26
Slide and Transparency 28
Computer and Drafting Table 29
8 Function Graphs 30

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9 CANCELLING PINCH POINTS: WIRE FRAMES 32
10 CANCELLING PINCH POINTS: HIDDEN LINES 33
11 CANCELLING PINCH POINTS: ELABORATIONS 34
12 SPINNING A PAIR OF PANTS 35
13 LAMINATION AND EIGHT KNOT 37
14 CABLE KNOT AND COMPANION 38
15 CABLING TEMPLATE 39
16 OCTAHEDRAL HYPERBOLIC MANIFOLD 40
17 TETRAHEDRAL HYPERBOLIC MANIFOLD 41
18 BORROMEAN ORBIFOLD 42

CHAPTER 3 PICTURES IN PERSPECTIVE 43

1 LINEAR PERSPECTIVE 44
2 HORIZON AND ZENITH 47
3 RECIPE FOR A CUBE 49
4 CUBE IN 3-2-1 POINT PERSPECTIVE 51
5 LOOKING FOR A CUBE 53
6 DIAGONALS 55
7 WHEEL AND AXLE 57
8 AXONOMETRICS 58
9 CHIAROSCURO 61

CHAPTER 4 THE IMPOSSIBLE TRIBAR 65

1 PENROSE TRIBAR 66
2 IMPOSSIBLE QUADRILATERAL 67
3 PIPELINE 69
4 DICE 70
5 POINCARÉ GROUP 71
6 CUBIC LATTICE 73
7 MAN IN A CUBE 74
8 LEFT HAND/RIGHT HAND 76

CHAPTER 5 SHADOWS FROM HIGHER DIMENSION 77

1 CROSSING A CHANNEL 79
2 PLÜCKER CONOID 81
3 PINCHING A ROMAN SURFACE 84
4 GAUSS MAP AND CROSS CAP 87
5 BOY SURFACE 90
6 SLICE AND SHADOW 93
7 WHITNEY BOTTLE 94
8 ROMBOY DEFORMATION 95

CHAPTER 6 SPHERE EVERSIONS 99

1 GOLDEN RECTANGLE 100
2 CROSS CAP AND HANDLE 101
3 FORBIDDEN EVERSION 103

4 BASEBALL MOVE 105
5 CHAPEAU DIAGRAM 108
6 ASTROID/DELTOID 109
7 ASSEMBLING IMMERSIONS 111
8 MORIN EVERSION 113
9 TOBACCO POUCHES 115
10 ASTROID FAMILY 118
11 CATASTROPHE MACHINE 121
12 MORIN TWIST 123
13 ORIGINAL DESIGN 124

CHAPTER 7 GROUP PICTURES 125

1 ZIPPING UP A DOUBLE TORUS 127
PRESENTING THE DOUBLE TORUS 129
2 BRAIDING HOMEOMORPHISM 130
3 ARTIN SWAP 132
4 SPHERICAL BRAIDS 133
5 PLATE TRICK 135
6 WHORL AND CHIMNEY 137
7 SWAPPING HANDLE CORES 140
8 SWAP DIAGRAM 141
9 LICKORISH TWIST 142
10 KING SOLOMON SEAL 145

CHAPTER 8 THE FIGURE EIGHT KNOT 149

1 PROJECTIONS OF THE KNOT 150
2 HEXAHEDRAL COMPLEMENT 152
3 HEXAHEDRAL GLUING DIAGRAM 154
4 SEIFERT SPANNING SURFACE 156
5 SIX HAKEN SURFACES 158
6 FIBER MNEMONIC 159
7 THE OWL AND THE PUSSYCAT 161
8 ISOTOPIC SEIFERT SURFACES 163
9 TRIVIAL FIBRATION 164
10 DOUBLE TORUS KNOT 165
11 TETRAHEDRAL EIGHT KNOT: ONE 167
12 TETRAHEDRAL EIGHT KNOT: TWO 168
13 TOROIDAL SHEAR ISOTOPY 170
14 CYLINDRICAL SHEAR ISOTOPY 171
15 COMPUTING THE MONODROMY 172
16 HOPF FIBRATION 175

POSTSCRIPT 176

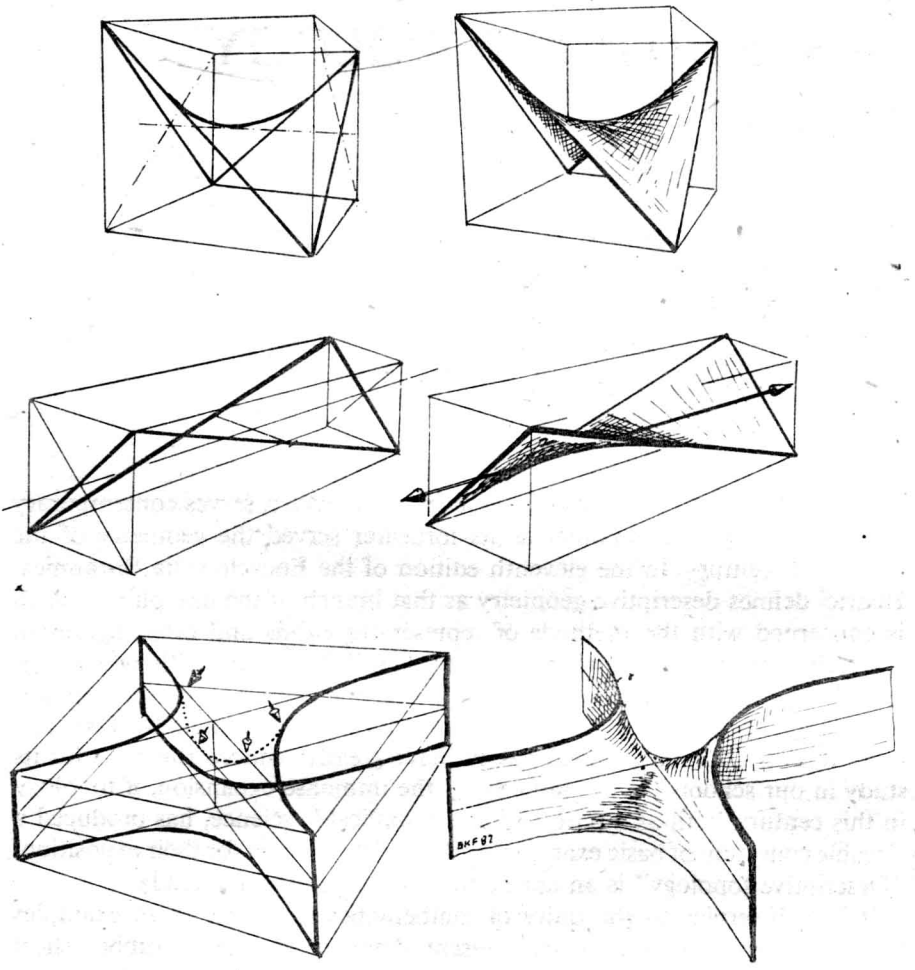
BIBLIOGRAPHY 180

INDEX 186

1
DESCRIPTIVE TOPOLOGY

Descriptive topology, namesake of *descriptive geometry*, serves contemporary topology in the same manner as its forbearer served the geometry of the nineteenth century. In the eleventh edition of the Encyclopedia Britannica, Henrici defines descriptive geometry as that branch of the discipline "which is concerned with the methods of representing solids and other figures in three dimensions by drawings in one plane." The rigor with which this subject used to be taught was justified by its application in the preparation of accurate architectural and engineering drawings. Photography first, and more recently, computer aided design, have greatly reduced the need for its study in our schools. At the same time, the immense expansion of topology in this century, both as a pure and as an applicable science, has produced a sizeable collection of basic examples and a variety of styles for their exposition. "Descriptive topology" is an apt name for their systematic study.

It is a disservice to the unity of mathematics to isolate these examples from their analytic origins and present them as so much "rubber sheet geometry." Since the days of Descartes, expressing geometrical information in that universal language of mathematics, algebra, has been immensely useful in the service of precision and economy of thought. Nevertheless, something is inevitably lost in this transcription. The task of descriptive topology is to unfold the visual secrets so often compressed into algebraic shorthand. To illustrate how this might be done sytematically, I shall draw on Whitney and Thom's theory concerning the generic forms of a surface extended in space or mapped into a plane. The graphical elements of my method are pictures of the canonical algebraic forms associated with surfaces in *general position*. These are *stable* in the sense that slight perturbations of their analytical formulation, or their physical expression, as on a computer screen, do not change them qualitatively. You will find a comprehensive introduction of the mathematics of stable mappings and their singularities in the text by Golubitsky and Guillemin [1973].



SADDLE IN A BOX

Figure 1

Let us start with the problem of graphing the *hyperbolic paraboloid* whose equation is $Z - XY = 0$. This saddle-shaped surface regularly taxes the artistic abilities of calculus teachers. Here are several ways to approach this problem. First, rewrite the equation to suggest a more imperative mood: *Let* $Z = XY$. This assigns to the Z-coordinate of a point the product of its XY-coordinates. Now treat this product asymmetrically by regarding the first factor as the parameter of a smooth transition between two skew lines in space. In other words, imagine the line $Z = tY$ in the plane $X = t$ turning about the X-axis as the plane moves parallel to itself in the X-direction. Figures 1(11) and 1(12) show such a saddle suspended inside a cube. Stretch the X-scale relative

to the other two and you produce a ribbon with a quarter twist as in 1(21) and 1(22). The abbreviation n(ij) refers to the detail in row i column j of figure n in the current chapter. If the figure referred to is in a different chapter, its number comes first. Thus the Möbius band in the lower right corner of Figure 9 in Chapter 6 is called 6:9(34).

A surface which can be regarded as the set of successive position of a curve moving in space is said to be *generated* by the curve. The utility of this notion in constructing a surface geometrically, in a picture or as a model is increased as the complexity of the generator and its motion is decreased. When the generator is a straight line, it is called a *ruled surface.* Since you can exchange X and Y in the above analysis, the hyperbolic paraboloid is generated by a line in two ways. It is a *doubly ruled surface.*

Reverse the equation like this: $XY = Z$ and you say that the point (X, Y) is constrained to lie on a hyperbola whose shape is controlled by the current value of Z. As Z moves from -1 to $+1$, the two branches of the hyperbola draw close to each other, touch in the shape of two crossing lines, and part again, fitting inside the asymptotes "the other way", see 1(31). Freezing this motion in space-time again generates a saddle similar to 1(32). Note the four places (arrows) where the contour of the object in the picture plane is tangent to curves on the surface.

SADDLE IN A DRUM. *Figure 2.*

Change to polar coordinates for another way of visualizing the hyperbolic paraboloid:

$$X = r\cos(t)$$
$$Y = r\sin(t)$$
$$Z = \tfrac{1}{2} r^2 \sin(2t) = a(t)\, r^2 = b(r)\sin(2t).$$

This defines two ways of generating the saddle depending on the order in which the r and t loops are nested. If t is taken for the parameter, the coefficient a of r^2 changes sinusoidally from -1 to $+1$ as t goes from 0 to π. Hence a vertical parabola changes its curvature from down to up as it makes a half turn about its axis of symmetry. Note that in the provisional line pattern, 2(11), there is an undesirable triple point. Can you detect how this coincidence was avoided when I traced 2(12) from this pattern?

If r is taken for the parameter then the scale coefficient b for a bent hoop on a cylinder, 2(21), goes to zero parabolically. The arrows point to important tangencies. The contour is marked as a dotted line. As the hoop shrinks towards the origin, it sweeps out the surface 2(22).

The mode in which analytical expressions and coordinate equations are formulated has considerable influence on the speed and precision with which the reader glimpses the same thing the writer means to describe. At times,

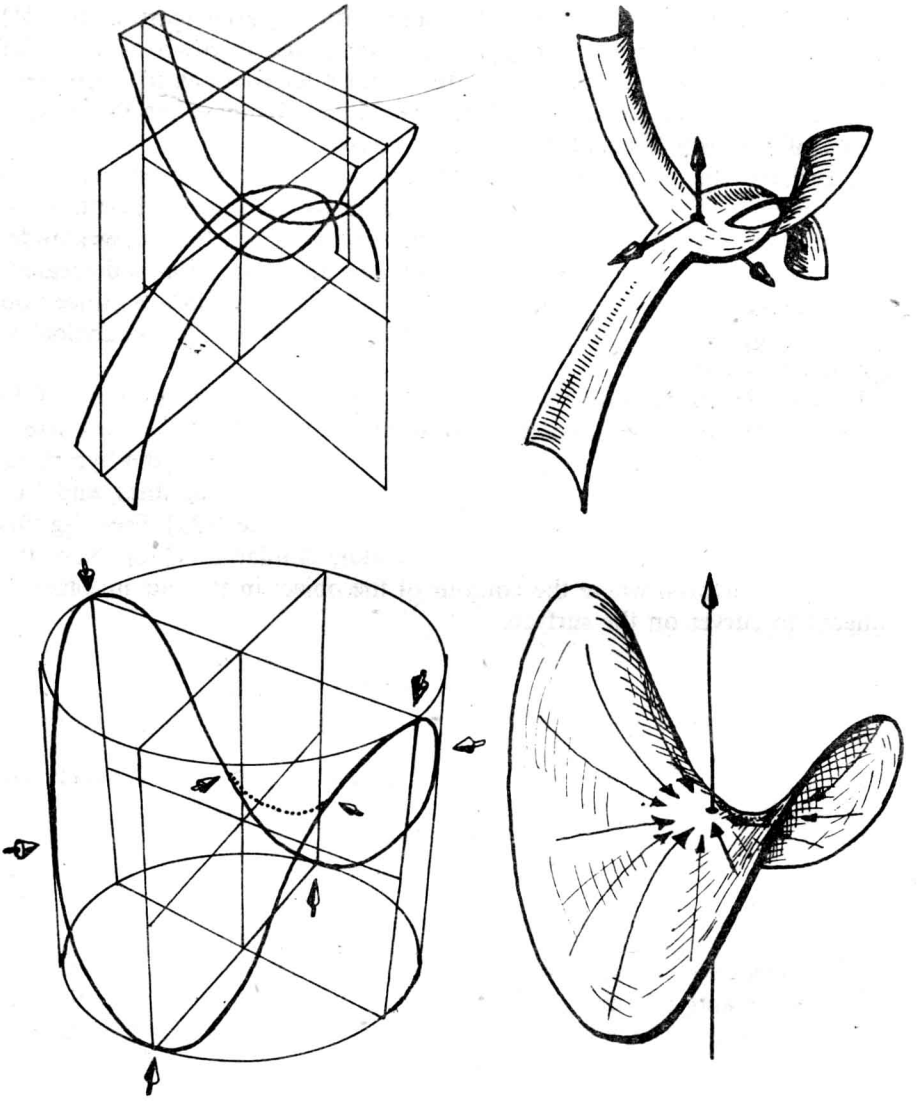


Figure 2 SADDLE IN A DRUM

efficiency requires a departure from customary style in analytical geometry. This is especially true for 3-dimensional objects and phenomena. Take, for example, the locus of points satisfying the polynomial equation $XY - Z = 0$. Unless your reader immediately recognizes the surface from this equation, or proceeds to rewrite the equation, he will probably imagine the following polygonal path: Go from the origin of the coordinate system to a point a

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