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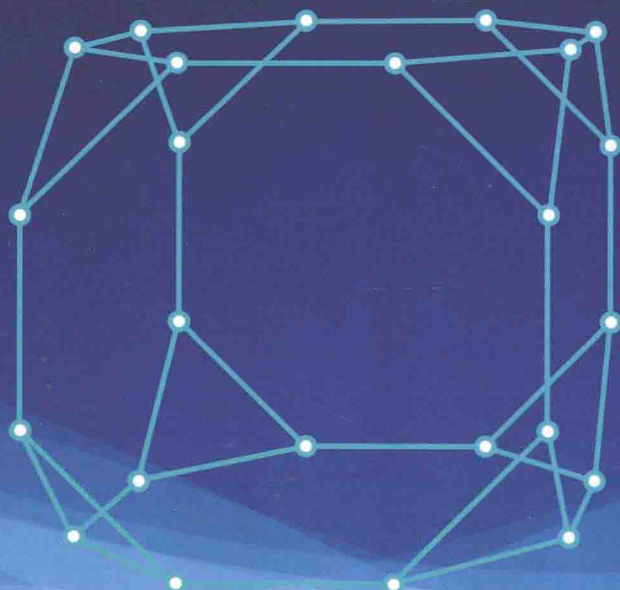
ADEX Theory

How the ADE Coxeter Graphs Unify Mathematics and Physics



Saul-Paul Sirag

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ADEX Theory

How the ADE Coxeter Graphs Unify
Mathematics and Physics

To Mary-Minn

Preface

This book is meant to be read by those who enjoy the interplay of mathematics and physics. Going back to the age of Archimedes, all of the advances in physics have been accomplished by a deep relationship between these two seminal fields of human knowledge. Indeed, many of the advances in mathematics have been inspired by attempting to solve problems in physics. Perhaps Newton is the prime exemplar of this approach. His attempt to understand gravity led to his development of the calculus, differential equations, and other areas of mathematics.

Physicists today are engaged in the very difficult, but enthralling, effort to unify all the forces. By far the most promising path seems to be string theory, which has evolved into a vast enterprise of mathematical discovery and application to the quantum theory of gravity unified with the Standard Model gauge forces.

For several decades now, I have been convinced that the organizing principle for this physical unification is the vast unification process afforded by the unification of mathematical structures inherent to the ADE classification of twenty-some very different mathematical objects. I have been impressed by the fact that all of these mathematical structures seem to have applications in physics — especially the physics of string theory.

Accordingly I have used the term ADEX theory to refer to the study and application of all the ADE-classified mathematical objects. The X should stand for the structure underlying all the ADE classified objects. Each of these mathematical objects is a separate window into the underlying structure. It seems that this underlying structure must be the quantum physical world as elaborated in string theory.

In developing my approach to string theory by way of ADEX theory, I have been encouraged, inspired, and instructed by many colleagues (including those who argued against my views), I would like to acknowledge especially.

Nick Herbert, Jeffrey Mishlove, Elizabeth Rauscher, Jack Sarfatti, Creon Levit, Tony Smith, and most of all, Louis Kauffman, who has over the years been prompting me to write this book.

Note to the Reader

For references in the text indicating papers and books to be found in the Bibliography, I have used the convention of square brackets with the authors name and date of publication, and occasionally a specific page number.

For example: [Yau, 1977], [Born, 1971, p. 158].

Some of these references occur also in the Glossary, whose terms I have described rather fully, rather than more briefly.

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Chapter 1

Introduction

In 1974, Arthur Young, at the Institute for the Study of Consciousness in Berkeley, California, asked me as his research associate to work out the symmetry group of a toy tetrahedron. This turned out to be $\mathcal{S}_4$, the symmetric-4 group, which is the set of all 24 permutations of four objects (for example, the four vertices of the tetrahedron). The rotations of the tetrahedron corresponded to the 12-element normal subgroup of $\mathcal{S}_4$, called $\mathcal{A}_4$, the alternating-4 group, or the tetrahedral group. The $\mathcal{S}_4$ group also includes reflections of the tetrahedron and is called the octahedral group because it corresponds to all the rotations of the octahedron (or the cube, which is its dual).

These group structures entail the basic themes of this book: symmetry, rotation, reflection, permutation, subgroup, duality, commutativity and non-commutativity, cosets, mappings, representations, and character tables. Underlying it all is the interplay between geometry and algebra.

Moreover, the tetrahedral ($\mathcal{T}$) and octahedral ($\mathcal{O}$) groups correpond to two of the ADE Coxeter graphs, E_6 and E_7, via the McKay correspondence groups $\mathcal{TD}$ and $\mathcal{OD}$, which are double covers of $\mathcal{T}$ and $\mathcal{O}$.

The ostensible connection between group structures and consciousness (a very controversial topic) was via Arthur Eddington's fascination with symmetry groups as fundamental to physics. In his book, *The Philosophy of Physical Science*, Eddington [1939] wrote:

"The recognition that physical knowledge is structural knowledge abolishes all dualism of consciousness and matter. Dualism depends on the belief that we find in the external world something of a nature incommensurable with what we find in consciousness; but all that physical science reveals to us in the external world is group-structure, and group-structure is also to be found in consciousness. When we take a structure of sensations in a particular consciousness and describe it in physical terms as part of the structure of an external world, it is still a structure of sensations. It would be entirely pointless to invent something else for it to be a structure of."

Since 1939 the importance of group theory in physics has grown by leaps and bounds, and is now at the very center of advances in unified field theory and other areas of theoretical physics. Indeed, prior theoretical advances such as Newtonian mechanics and Maxwell's electromagnetic theory are viewed from the standpoint of symmetry groups.

By far the deepest theoretical advance afforded by the group-theory approach is the set of ADE Coxeter graphs, which originally classified the most important (and useful) finite reflection groups, and in the form of Dynkin diagrams classified the most important (and useful) Lie groups. This work of the 1930s and 1940s has, in the last several decades evolved into the ADE classification of twenty-some mathematical categories, due to the work of many other mathematicians. The Russian mathematician, V. I. Arnold, as one of the most active and perceptive theoreticians in this process, wrote in his book, *Catastrophe Theory* [Arnold, 1986]:

"At first glance, functions, quivers, caustics, wave fronts and regular polyhedra have no connection with each other. But in fact, corresponding objects bear the same label not just by chance: for example, from the icosahedron one can construct the function $x^2+y^3+z^5$, and from it the diagram E_8, and also the caustic and wave front of the same name.

"To easily checked properties of one of a set of associated objects correspond properties of the others which need not be evident at all. Thus the relations between all the A, D, E-classifications can be used for the simultaneous study of all simple objects, in spite of the fact that the origin of many of these relations (for example, of the connections between functions and quivers) remains an unexplained manifestation of the mysterious unity of all things."

It is especially striking that the objects classified by the ADE graphs are of great utility in the advance of unified field theory afforded by superstring theory and its generalization to M-theory. The list of ADE classified categories (to be described in this book) should make this plain.

Lie algebras (and Lie groups): gauge group theory;

Kac–Moody (infinite-d) algebras;

Coxeter (reflection) groups, also called Weyl groups;

Coxeter arrangements;

Klein–DuVal singularities;

McKay correspondence groups (finite subgroups of $\mathcal{SU}(2)$);

Hyperspace crystallography;

Sphere-packing lattices (root lattices): error-correcting codes;

Quantizing lattices (weight lattices): analog/digital transforms;

Conformal field theories (living on the 2D string worldsheet);

Gravitational instantons (cf. Penrose twistors); ALE spaces;

Thom–Arnold catastrophe structures;

Heisenberg algebras (in various hyperspaces);

Generalized braid groups (cf. knots and links);

Quivers.

It is quite astonishing that this great diversity of categories should be classified by the simplest possible graphs.

Here is the complete list of ADE Coxeter graphs (and some salient correspondences) [Coxeter, 1973; McKay, 1980]:

Coxeter graph	McKay group	Coxeter number → Lie group dimension
A_n o-o-...-o (n nodes)	$\mathcal{Z}_{n+1}$	$n+1 \to n^2 + 2_n$
D_n o-o-...-o / | / o (n nodes) $n > 3$	$\mathcal{Q}_{n-2}$	$2n-2 \to 2n^2 - n$
E_6 o-o-o-o-o / | / o	$\mathcal{TD}(24)$	$12 \to 78$
E_7 o-o-o-o-o-o / | / o	$\mathcal{OD}(48)$	$18 \to 133$
E_8 o-o-o-o-o-o-o / | / o	$\mathcal{ID}(120)$	$30 \to 248$

$\mathcal{TD}(24)$ means $\mathcal{TD}$ has 24 elements, etc.

Note that the Lie group dimension is $nk + n$, where n is the rank of the Coxeter graph, and k is the Coxeter number. This is just a taste of many relationships afforded by the structure of the ADE Coxeter graphs.

One striking feature of these Coxeter graphs is that, while there are the two infinities of A graphs and D graphs, there are only three E graphs: of ranks 6, 7 and 8. Moreover, these graphs correspond to the five Platonic solids:

$E_6 \to$ tetrahedron (self dual)

$E_7 \to$ octahedron (dual to the cube)

$E_8 \to$ icosahedron (dual to the dodecahedron).

Thus there are three Platonic symmetry groups corresponding to the three E-type graphs. It is plausible to think of the ADE graphs as the ultimate Platonic archetypes.