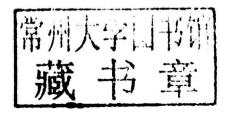
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A BRIEF HISTORY OF NUMBERS

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A Brief History of Numbers



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in subsequent works of fellow historians. Rather, it is a work of synthesis meant to provide a broad overview of what historians have written and to direct prospective readers to their works. I do not claim, however, to have reflected all the existing views on all the topics, nor all of the divergent and sometimes opposed interpretations that historians have suggested for the various episodes that I discuss. At the same time, given the style of the book, I did not want to fatigue the reader with a full scholarly apparatus to support each and every one of the claims I have made in the text. Direct references are only to texts that are cited as full paragraphs, or to texts that elaborate in greater detail on issues that I have only mentioned in passing but that, for reasons of space, I have left as brief remarks. Still, I have provided at the end of the book a somewhat detailed list of texts for further reading, organized more or less in accordance with the chapters of the book. This is my way to explicitly acknowledge the works on which I have directly relied for the various issues discussed. This is my way to indicate my great intellectual debt to each of these authors. Readers interested in broadening the scope of my account will find in those texts large amounts of additional illuminating material. The texts also provide direct references to the primary sources, which lend support to the historical claims that I put forward in this book. In fact, in consideration with space limitations, except in cases where I have cited directly, I do not include primary sources in the bibliographical list, in the assumption that interested readers can easily find detailed information either in the secondary sources or collections cited or by Internet search.

One typical reader that I have had in mind when writing the book is an undergraduate student in mathematics. As she is struggling to develop demanding technical skills in the courses that she is now studying, a book like this may help her make further sense, from a perspective that is different from that found in textbooks and in class, of the world of ideas that she is gradually becoming acquainted with. Her apprenticeship, I think, may gain in depth and richness by reading a book like this one. A second, typical kind of reader I have envisaged for the book is a teacher of mathematics. This book, I believe, may contribute to his efforts in bringing to the classroom a broader understanding of his own discipline, with an eye on the long-range historical processes that have been at play in shaping it.

I have also tried to write in such a way as to allow for a wider readership that would include intellectually curious high-school pupils with an inclination for mathematics, professional mathematicians, scientists and engineers, and other educated readers of various kinds. Hence I have tried to strike a natural balance between technicalities and broad historical accounts. In this respect, there are slight differences among the various chapters. Some are more technically demanding, some less so. In all cases, however, I have made my best efforts to write concisely and in a way that will be within the reach of readers of various backgrounds and will elicit their interest.

Still, I have certainly not tried to reach every possible readership. While writing, I have often had in mind a famous statement of Stephen Hawking in the preface to his Brief History of Time (from which I shamelessly took inspiration in choosing my own title). He declared that he had followed the advice of his publisher to the effect that he should not include too many formulae in the book, since each individual formula would reduce the number of potential readers by a half. Hawking wisely followed this advice and he included in the book only one formula, the inevitable $e = mc^2$. In retrospect, this was certainly a very good piece of advice, as the book turned into an unprecedented best-seller and it was also translated into an amazingly large number of languages.

I can only wish for such a success at the box-office with my book, but I have to confess that I have had to include more than one formula. Actually, the reader will find here many formulae and diagrams, though, in general, they are not particularly complicated. I think that anyone who thought that the topic of the book is appealing for her, in the first place, will not be deterred by what I have included. And I simply could not leave these formulae and diagrams out of my account, if I wanted to speak seriously about the history of numbers. Still, in places where I have thought that some readers would like to see further technical details, while others would find these details to be a hindrance to their understanding. I have relegated them to a separate appendix in the respective chapter. These appendices can be skipped without fear of loss of continuity in understanding the main text, but I strongly suggest that the reader should not do that and should rather devote the necessary effort where needed. I think that such effort will be rewarded. In some places, I have provided some detailed proofs or technical explanations in the main text, assuming that they will be of interest to most readers and that they are necessary for achieving the full picture that I am trying to convey.

Several friends and colleagues have been kind enough to read parts of the book and to offer me their useful critical comments. They have helped me improve the contents, the structure and the style of my text to a considerable extent. I thank them sincerely: Alain Bernard, Sonja Brentjes, Jean Christianidis, Michael Fried, Veronica Gavagna, Jeremy Gray, Niccolò Guicciardini, Albrecht Heeffer, Victor Katz, Jeffrey Oaks and Roy Wagner. I also acknowledge the wise advise of three anonymous referees.

It is a pleasure to thank Keith Mansfield and his team at OUP (Clare Charles and Daniel Taber) for their help and editorial support. My sincere thanks go to Mac Clarke for a superb job as copy editor of my text. Likewise, I want to thank Kaarkuzhali Gunasekaran and her team at Integra Software Services for their highly professional approach in typesetting this book.

Special thanks go to my dear friend Lior Segev for having introduced me into the basics of LaTeX. I had been able thus far, somehow, to avoid doing this, but, in retrospect, I am glad to have learned the craft now under his guidance.

I am always happy to acknowledge the continued and friendly support of Barbara and Bertram Cohn, as the proud incumbent of the Chair in History and Philosophy of Exact Sciences at Tel Aviv University that bears their name.

Finally, I thank my dear family for their unconditional support and for their blind trust that whatever I do, it must surely be important and praiseworthy. As always, I have done my best efforts to stand up to their high expectations.

Tel Aviv University, May 2015

Leo Corry



CONTENTS

1.	1. The System of Numbers: An Overview						
	1.1 From natural to real numbers	3					
	1.2 Imaginary numbers	3					
	1.3 Polynomials and transcendental numbers	11					
	1.4 Cardinals and ordinals	15					
2.	2. Writing Numbers—Now and Back Then						
	2.1 Writing numbers nowadays: positional and decimal	17					
	2.2 Writing numbers back then: Egypt, Babylon and Greece	24					
3.	Numbers and Magnitudes in the Greek Mathematical Tradition	31					
	3.1 Pythagorean numbers	32					
	3.2 Ratios and proportions	35					
	3.3 Incommensurability	39					
	3.4 Eudoxus' theory of proportions	42					
	3.5 Greek fractional numbers	45					
	3.6 Comparisons, not measurements	47					
	3.7 A unit length	50					
	Appendix 3.1 The incommensurability of $\sqrt{2}$. Ancient and modern proofs						
	Appendix 3.2 Eudoxus' theory of proportions in action						
	Appendix 3.3 Euclid and the area of the circle						
4.	Construction Problems and Numerical Problems						
	in the Greek Mathematical Tradition						
	4.1 The arithmetic books of the <i>Elements</i>	64					
	4.2 Geometric algebra?	66					
	4.3 Straightedge and compass	67					
	4.4 Diophantus' numerical problems	71					
	4.5 Diophantus' reciprocals and fractions	78					
	4.6 More than three dimensions	80					
Appendix 4.1 Diophantus' solution of Problem V.9 in Arithmetica							

5. Numbers in the Tradition of Medieval Islam						
	5.1 Islamicate science in historical perspective	88				
	5.2 Al-Khwārizmī and numerical problems with squares	90				
5.3 Geometry and certainty						
	5.4 Al-jabr wa'l-muqābala	97				
	5.5 Al-Khwārizmī, numbers and fractions	100				
	5.6 Abū Kāmil's numbers at the crossroads of two traditions	103				
	5.7 Numbers, fractions and symbolic methods	107				
	5.8 Al-Khayyām and numerical problems with cubes	111				
	5.9 Gersonides and problems with numbers	116				
	Appendix 5.1 The quadratic equation. Derivation of the algebraic formula	120				
	Appendix 5.2 The cubic equation. Khayyam's geometric solution	121				
6.	Numbers in Europe from the Twelfth to the Sixteenth Centuries	125				
	6.1 Fibonacci and Hindu-Arabic numbers in Europe	128				
	6.2 Abbacus and coss traditions in Europe	129				
	6.3 Cardano's Great Art of Algebra	138				
	6.4 Bombelli and the roots of negative numbers	146				
	6.5 Euclid's <i>Elements</i> in the Renaissance	149				
	Appendix 6.1 Casting out nines	150				
7.	Number and Equations at the Beginning of the Scientific Revolution	155				
	7.1 Viète and the new art of analysis	157				
	7.2 Stevin and decimal fractions	163				
	7.3 Logarithms and the decimal system of numeration	167				
	Appendix 7.1 Napier's construction of logarithmic tables	171				
8.	Number and Equations in the Works of Descartes, Newton					
and their Contemporaries						
	8.1 Descartes' new approach to numbers and equations	176				
	8.2 Wallis and the primacy of algebra	182				
	8.3 Barrow and the opposition to the primacy of algebra	187				
	8.4 Newton's Universal Arithmetick	190				
	Appendix 8.1 The quadratic equation. Descartes' geometric solution Appendix 8.2 Between geometry and algebra in the seventeenth	196				
	century: The case of Euclid's Elements	198				
9.	New Definitions of Complex Numbers in the Early					
	Nineteenth Century	207				
	9.1 Numbers and ratios: giving up metaphysics	208				
	9.2 Euler, Gauss and the ubiquity of complex numbers	209				
	9.3 Geometric interpretations of the complex numbers	212				
	9.4 Hamilton's formal definition of complex numbers	215				
	9.5 Beyond complex numbers	217				
	9.6 Hamilton's discovery of quaternions	220				

10. "What Are Numbers and What Should They Be?"			
Understanding Numbers in the Late Nineteenth Century	223		
10.1 What are numbers?	224		
10.2 Kummer's ideal numbers	225		
10.3 Fields of algebraic numbers	228		
10.4 What should numbers be?	231		
10.5 Numbers and the foundations of calculus	234		
10.6 Continuity and irrational numbers	237		
Appendix 10.1 Dedekind's theory of cuts and Eudoxus' theory of			
proportions	243		
Appendix 10.2 IVT and the fundamental theorem of calculus	245		
11. Exact Definitions for the Natural Numbers: Dedekind,			
Peano and Frege	249		
11.1 The principle of mathematical induction	250		
11.2 Peano's postulates	251		
11.3 Dedekind's chains of natural numbers	257		
11.4 Frege's definition of cardinal numbers	259		
Appendix 11.1 The principle of induction and Peano's postulates	262		
12. Numbers, Sets and Infinity. A Conceptual Breakthrough at			
the Turn of the Twentieth Century	265		
12.1 Dedekind, Cantor and the infinite	266		
12.2 Infinities of various sizes	269		
12.3 Cantor's transfinite ordinals	277		
12.4 Troubles in paradise	280		
Appendix 12.1 Proof that the set of algebraic numbers is countable	287		
13. Epilogue: Numbers in Historical Perspective	291		
References and Suggestions for Further Reading	295		
Name Index			
Subject Index			

CHAPTER 1

The System of Numbers: An Overview

athematics and history, history and mathematics. One can hardly think of two fields of knowledge that are more different from each other—some would say outright opposed—in both their essence and their practice.

At its core, mathematical knowledge deals with *certain*, *necessary* and *universal* truths. True mathematical statements do not depend on contextual considerations, either in time or in geographical location. Generally speaking, established mathematical statements are considered to be beyond dispute or interpretation.

The discipline of history, on the contrary, deals with the *particular*, the *contingent* and the *idiosyncratic*. It deals with events that happened in a particular location at a particular point in time, and that happened in a certain way but could have happened otherwise. Historical statements are always partial, debatable and open to interpretation. Arguments put forward by historians keep changing with time. "Thinking historically" and "thinking mathematically" are clearly two different things.

But, if by "thinking historically" about whatever topic, we mean anything other than just establishing a chronology of facts, then an interesting question is whether we can think historically about the ways in which people have been "thinking mathematically" throughout history, and about the processes of change that have affected these ways of thinking. If mathematics deals with universal truths, how can we speak about mathematics from a historical perspective (other than establish the chronology of certain discoveries)? What is it that changes through time in a discipline that is, apparently, eternal?

This is precisely what this book is about: a brief historical account (which is not just a chronology) of how people have thought about numbers in changing historical circumstances. Not just *what* they knew about numbers, but also, and mainly, *how* they knew it and what did they think *about* what they knew. The kind of questions we will be pursuing here include the following:

 What were the basic concepts around which knowledge about numbers and about their properties was built in different historical contexts?

- How did it happen that these concepts changed through time?
- What were the main mathematical problems that required, in a certain historical context, the use of numbers of various kinds?
- How were numbers written in different cultures and how did different mathematical notations either help or hinder further developments in the conceptions of numbers and in the techniques of calculation?
- What was the relationship, in the various mathematical cultures, between arithmetic and other, neighboring, disciplines (mainly geometry), and what were the philosophical conceptions of the mathematicians involved in arithmetical activity?
- How did practical considerations encourage (or sometimes discourage) the adoption of certain ideas pertaining to numbers?
- What roles did the institutions of knowledge that developed in the various cultures play in promoting or preventing the development of a given conception about numbers?

Numbers are important in our world. The world around us is saturated with them. Numbers appear routinely and increasingly not only in scientific and technological contexts, but also in the news and in commerce, as well as in many aspects of our private life. The vernacular of natural science—and particularly of physics—is mathematics, at the heart of which lies numbers. Also in the social sciences-economics and political science above all—the language of numbers is central to much of both its theorizing and its empirical work. In public life, numbers are not just ubiquitous as a tool for explanation, but also a necessary means for administration and control. The main tools used nowadays in bureaucratic systems all around the world rely on the processing of data and (sometimes manipulative) analysis of numbers. The centrality of digital computers in all aspects of life just emphasizes and makes even more manifest the truth of these basic facts.

Numbers are present in so many aspects of our day-to-day life that we take their presence for granted. But this situation is by no means an inevitable law of nature. Rather it is the result of a very specific, long, convoluted and multilayered historical process. A very important turning point in this process came in the seventeenth century, in the framework of what is typically known as the "scientific revolution." Developments in disciplines such as the science of mechanics during this period turned them into all-out mathematical branches, in opposition to what was the case in the Aristotelian tradition that had dominated European intellectual life since the late scholastic period around the fourteenth century. The dominant ideal of explanation of natural phenomena before the seventeenth century did not encourage (and indeed sometimes actively opposed) the search for mathematically formulated laws of nature. During the eighteenth and nineteenth centuries, the role of numbers became increasingly central in natural science as well as in other aspects of knowledge and day-to-day life.

These kinds of significant transformations that affected the role of numbers in society throughout history have attracted the attention of historians, and, indeed, many educated audiences are well aware of them to various degrees of detail. What is less recognizable at first sight, and has typically escaped the attention of those same audiences, is that even the very idea of what is a number and how it is used within mathematics has

itself been at the focus of a more circumscribed yet no less long and complex process of debate, evolution and continuous modification. This book focuses on this process, a process that is historically important yet subtle and rather inconspicuous within the overall accepted conceptions about the development of science.

Of course, anyone who gives some thought to it will not fail to realize that all of our current knowledge about numbers and about their properties was achieved as the accumulated product of the efforts of generations of mathematicians who devoted great intellectual energies to it. But, more than with any other field of knowledge, this process tends to be conceived as essentially linear and straightforward. More than with any other field of knowledge, the role of the historian reconstructing this process tends to be seen as that of a chronologist whose main task is just to determine who did what for the first time and when.

Typically, the *process* leading to our present conceptions about numbers and to our knowledge of their properties is presumed to be clear, historically unproblematic and unsurprising, except perhaps for the dates and names to be attached to each step. The idea of a truly *historical process*—where individuals and groups are faced with dilemmas and need to make real choices between alternatives, where wrong alleys, detours and dead-ends are sometimes followed for long periods of time, where contrary views are held by parties who are equally knowledgeable within their subjects—seems to many to be foreign to the development of mathematics and in particular to the development of ideas related to numbers.

I intend to show that the history of numbers is an intriguing story that developed at various levels, in a surprisingly non-linear fashion, involving many unexpected moves, dead-ends and also, of course, highly ingenious ideas and far-reaching successes. The outcome of the story is the creation of a beautifully conceived world of numbers, as it crystallized by the turn of the twentieth century. Since then, some new ideas and refinements of the existing ones have been added, but our basic conception of the system of numbers and how it is built was essentially attained by that time, after centuries of important breakthroughs accompanied by hesitations, misunderstandings and uncertainties.

In order to make this account more perspicuous and effective, I have chosen to begin with an introductory chapter that is technical and non-historical. It is intended to offer a general overview of the system of numbers such as it is conceived nowadays. This includes a general description of the various kinds of numbers and the relationships between them, the accepted ways to write these numbers, and the assumptions underlying them. I have also included some truly basic results about numbers. Some prospective readers of this book will surely be acquainted with the material discussed in this chapter, but it is convenient to provide a common basic language with which to proceed to the historical account that will appear in subsequent ones.

1.1 From natural to real numbers

The very systematic conception of the world of numbers that is accepted nowadays envisages a carefully constructed hierarchy of various classes of increasing complexity that starts with the natural numbers and gradually adds new types—negative, rational,

irrational—up to the real numbers. This view was attained only by the turn of the twentieth century. If we consider earlier periods, such as between the seventeenth and nineteenth centuries, in which so much important scientific progress was achieved in Europe, particularly in mathematics and in mathematical physics, it may come as a great surprise that the concept of number was still quite confused and lacked anything like a proper foundation. Remarkably, this did not substantially hinder the enormous progress in those disciplines where mathematics was now playing a central role.

This in itself is a non-trivial historical phenomenon that deserves closer attention. Indeed, there exists an interesting and not always properly emphasized gap between the actual, rather erratic, historical evolution of mathematical ideas, on the one hand, and, on the other hand, the subsequent, neat textbook presentation of these ideas as part of a perfectly structured body of knowledge. Students of mathematics typically learn in their university courses, say of calculus, a very tidy, comprehensive and well-arranged picture of the discipline, where each stage of the gradual presentation of results seems to arise naturally and smoothly from the previous one. Historically speaking, in contrast, these ideas evolved in a much more chaotic, unordered and unexpected way. As a matter of fact, sometimes they evolved in a succession that is the exact opposite to how they are conveniently presented in retrospect.

In the case of the calculus, for example, the concept of limit, typically learnt in the early lessons of any course in analysis, only arose at the beginning of the nineteenth century, after many of the basic techniques of calculation with derivatives and integrals and of solving differential equations had already been developed. And in turn, in order to provide a truly well-elaborated, formal definition of limit, it was necessary to come up with a much clearer idea of the foundations and basic structure of the system of real numbers, which became available only at the end of the nineteenth century. This, in turn, required a much better knowledge of the concept of set, which started to be developed only around this time and became well understood only in the first decade of the twentieth century.

The most basic idea related to the concept of number is that of counting. We associate this idea with the concept of *natural* numbers, namely those appearing in the series 1, 2, 3, 4, 5, . . . This is the starting point of arithmetic both in the *cognitive* sense (i.e., the way in which any child begins to learn numbers) and in the *historical* sense (i.e., arithmetical knowledge, in whatever culture, starts from knowledge about the natural numbers). This may seem an obvious point, but it is far from being so. As already mentioned, these two lines of development – cognitive and historical – typically differ, especially when it comes to the more advanced aspects of mathematics. We will see many important examples of this in what follows. But here at very the basis of arithmetic, they essentially coincide.

It is customary to denote the collection of natural numbers by the letter \mathbb{N} , as follows:

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

An important issue that arises immediately in connection with the natural numbers is the special role accorded to the prime numbers within this collection. A prime number is usually defined as a natural number having only two exact divisors: itself and 1 (and

4 THE SYSTEM OF NUMBERS: AN OVERVIEW