NONLINEAR
PHYSICAL
SCIENCE

Vladimir V. Uchaikin

Fractional Derivatives for Physicists and Engineers

Volume II Applications

物理及工程中的分数维微积分

第II卷 应用

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Wuli Ji Gongcheng Zhong De Fenshuwei Weijifen

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With 59 figures

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Author
Vladimir V. Uchaikin
Ulyanovsk State University
L. Tolstoj str 42
Ulyanovsk 432 970, Russia
Email: uchaikin@sv.uven.ru

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NONLINEAR PHYSICAL SCIENCE

非线性物理科学

NONLINEAR PHYSICAL SCIENCE

Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

- New findings and discoveries in nonlinear physics and mathematics
- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

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7.1.1 Non-relativistic case

We start reviewing applications of fractional calculus with the following mechanical problem.

A classical, non-relativistic, point particle of mass m with potential energy $U = mgy$ begins to slide without friction along a curve running through a vertical plane $x-y$ to the origin and reaches it at time τ (Fig. 7.1). The problem is to find the function $\tau(h)$ that specifies the total time of descent from an initial height h. A special case of the problem when $\tau(h) = \text{const}$ is called the *tautochrone problem* (from Greek prefixes *tauto* meaning "same" and *chrono* "time").

The principle of conservation energy says

$$\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = mg(h-y),$$

where s is the length of the curved segment between the origin and a current position, and $h-y$ is the descent of the particle during time t. Hence

$$dt = -\frac{ds}{\sqrt{2g(h-y)}}.$$

After integrating, we directly arrive at the equation

$$\tau(h) = \int_0^h \frac{ds/dy}{\sqrt{2g(h-y)}}dy = \sqrt{\frac{\pi}{2g}}\ {}_0^{1/2}\mathsf{D}_h s(h)$$

called *Abel's integral equation*, solution of which has opened a way to the land of fractional equations (Abel, 1881).

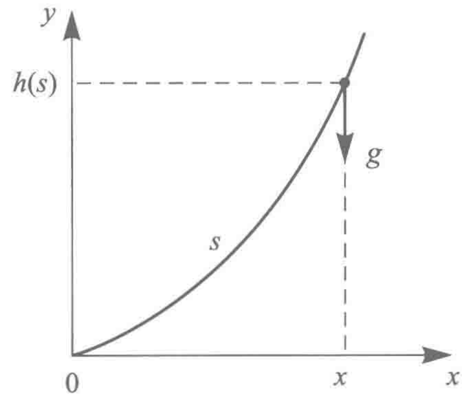

Fig. 7.1 Abel's mechanical problem.

Muñoz and Fernandéz-Anaya (2010) apply the fractional approach to investigation of the properties of the tautochrone and brachistochrone curves by introducing a family of curves complying with relations where the time of descent is proportional to a fractional power of the height difference.

7.1.2 Relativistic case

The relativistic counterpart of this problem has been studied by Kamath (1992). The methods of fractional calculus are shown to be more useful in the derivation of the exact relativistic tautochrone. Relativistic kinematics yields the equation of conservation energy

$$mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} + Q,$$

where Q is the energy lost by the particle from the gravitation field as it is released from a height h and is given by

$$Q = mc^2\left\{1-\exp[g(h-y)/c^2]\right\}$$

(Goldstein and Bender, 1986). From two these equations, we find

$$v(y) = c\sqrt{1-\exp[2g(y-h)/c^2]}.$$

The time of fall is

$$T = \int_0^T dt = -\int_h^0 \frac{ds}{v(y)} = \int_0^h \frac{s'(y)dy}{c\sqrt{1-\exp[2g(y-h)/c^2]}},$$

with $s'(y) = ds/dy$ being the arclength along the path joining the initial $(x_0, y_0 = h)$ and final $(0,0)$ end points. By rewriting this equation as

$$cT = i\int_0^h \frac{e^{-gy/c^2} s'(y)dy}{\sqrt{e^{-2gh/c^2} - e^{-2gy/c^2}}} = -\frac{ic^2}{2g}\int_0^h \frac{(-2g/c^2)e^{-gy/c^2}}{\sqrt{e^{-2gh/c^2} - e^{2gy/c^2}}} s'(y)dy,$$

one arrives at the fractional equation for the function $\eta(h) = s'(h)e^{gh/c^2}$ determining the sought curve:

$$c\sqrt{\pi}\, {}_0\mathsf{D}^{-1/2}_{\mu(h)}\eta(h) = 2iTg,$$

with $\mu(h) = e^{2gh/c^2} - 1$. Converting the equation to

$${}_0\mathsf{D}^{1/2}_{\mu(h)}\, {}_0\mathsf{D}^{-1/2}_{\mu(h)}\eta(h) = {}_0\mathsf{D}^{1/2}_{\mu(h)}(2iTg/c\sqrt{\pi}),$$

and using the composition rules, the author reduces it to the form

$$\sqrt{\pi}e^{gh/c^2}s'(h) = \frac{2iTg}{c\sqrt{\pi}}\frac{d}{d\mu(h)}\int_0^h \frac{-2ge^{-2gy/c^2}dy}{c^2\sqrt{e^{-2gh/c^2} - e^{-2gy/c^2}}}$$

$$= 2\frac{2iTg}{c\sqrt{\pi}}\frac{d[\mu(h)]^{1/2}}{d\mu(h)} = \frac{2iTg}{c\sqrt{\pi}}[\mu(h)]^{-1/2}.$$

Solution of this equation leads to the following parametric representation of the sought tautochrone:

$$e^{2gy/c^2} = 1 + \left(\frac{2Tg}{\pi c}\right)^2 \cos^2\theta,$$

$$\frac{gx}{c^2} = \theta - \frac{\pi}{2} + a\left(\frac{\pi}{2} - \arctan\left(\frac{1}{a}\tan\theta\right)\right)$$

with

$$a = 1 + \left(\frac{2Tg}{\pi c}\right)^2.$$

The non-relativistic tautochrone problem was generalized to an arbitrary potential $U(y)$ as well (Gómez and Marquina, 2008). In this case

$$dt = -\frac{ds}{\sqrt{(2/m)U(h-y)}}$$

and

$$T = -\int_{y_0}^{y} \frac{s'(y)dy}{\sqrt{(2/m)[U(y_0) - U(y)]}}. \tag{7.1}$$

Making $z = U(y)$, one can write

$$\frac{ds}{dy}dy = \frac{ds}{dz}dz$$

with

$$dz = U'dy,$$

so then Eq. (7.1) takes the form

$$T = -\sqrt{\frac{m}{2}} \int_{z_0}^{z} \frac{s'(z)dz}{\sqrt{z_0 - z}}.$$

Solution of this equation

$$x = \int_0^y \sqrt{\frac{2T^2}{m\pi^2} \frac{U'^2}{U} - 1}\, dy$$

was found in (Flores and Osler, 1999).

7.2 Inverse problems

7.2.1 Finding potential from a period-energy dependence

Let us look at Sect. 12 of "Mechanics" (Landau and Lifshitz, 1981) devoted to finding of potential energy $U(x)$ from the oscillation period T given as a function of the total energy E, $T = T(E)$ (Fig. 7.2). Starting as before from the principle of energy conservation, they obtain the integral equation

$$T(E) = 2\sqrt{2m} \int_0^{X(E)} \frac{dx}{\sqrt{E - U(x)}}, \tag{7.2}$$

where $X(E)$ is a root of equation $U(x) = E$ (for simplicity let the potential be an even function, monotonically increasing with moving from the origin). Passage to integration variable U leads to the fractional differential equation

$$T(E) = 2\sqrt{2m} \int_0^E \frac{dX(U)}{dU} \frac{dU}{\sqrt{E - U}} = 2\sqrt{2m\pi}\ {}_0^{1/2}\mathrm{D}_E X(E),$$

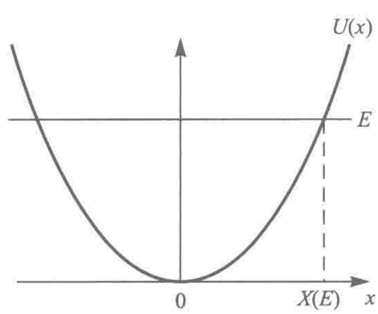

Fig. 7.2 Illustration to Eq. (7.2).