THIRD EDITION

FRACTAL GEOMETRY

Mathematical Foundations and Applications

Kenneth Falconer

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Third Edition

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Fractal Geometry

Preface to the first edition

I am frequently asked questions such as 'What are fractals?', 'What is fractal dimension?', 'How can one find the dimension of a fractal and what does it tell us anyway?' or 'How can mathematics be applied to fractals?' This book endeavours to answer some of these questions.

The main aim of the book is to provide a treatment of the mathematics associated with fractals and dimensions at a level which is reasonably accessible to those who encounter fractals in mathematics or science. Although basically a mathematics book, it attempts to provide an intuitive as well as a mathematical insight into the subject.

The book falls naturally into two parts. Part I is concerned with the general theory of fractals and their geometry. Firstly, various notions of dimension and methods for their calculation are introduced. Then geometrical properties of fractals are investigated in much the same way as one might study the geometry of classical figures such as circles or ellipses: locally, a circle may be approximated by a line segment, the projection or 'shadow' of a circle is generally an ellipse, a circle typically intersects a straight line segment in two points (if at all) and so on. There are fractal analogues of such properties, usually with dimension playing a key role. Thus, we consider, for example, the local form of fractals and projections and intersections of fractals.

Part II of the book contains examples of fractals, to which the theory of the first part may be applied, drawn from a wide variety of areas of mathematics and physics. Topics include self-similar and self-affine sets, graphs of functions, examples from number theory and pure mathematics, dynamical systems, Julia sets, random fractals and some physical applications.

There are many diagrams in the text and frequent illustrative examples. Computer drawings of a variety of fractals are included, and it is hoped that enough information is provided to enable readers with a knowledge of programming to produce further drawings for themselves.

It is hoped that the book will be a useful reference for researchers, by providing an accessible development of the mathematics underlying fractals and showing how it may be applied in particular cases. The book covers a wide variety of mathematical ideas that may be related to fractals and, particularly in Part II, provides a flavour of what is available rather than exploring any one subject in too much detail. The selection of topics is to some extent at the author's whim – there are certainly some important applications that are not included. Some of the material dates back to early twentieth century, whilst some is very recent.

Notes and references are provided at the end of each chapter. The references are by no means exhaustive; indeed, complete references on the variety of topics covered would fill a large volume. However, it is hoped that enough information is included to enable those who wish to do so to pursue any topic further.

It would be possible to use the book as a basis for a course on the mathematics of fractals, at postgraduate or, perhaps, final-year undergraduate level, and exercises are included at the end of each chapter to facilitate this. Harder sections and proofs are marked with an asterisk and may be omitted without interrupting the development.

An effort has been made to keep the mathematics to a level that can be understood by a mathematics or physics graduate and, for the most part, by a diligent final-year undergraduate. In particular, measure theoretic ideas have been kept to a minimum, and the reader is encouraged to think of measures as 'mass distributions' on sets. Provided that it is accepted that measures with certain (intuitively almost obvious) properties exist, there is little need for technical measure theory in our development.

Results are always stated precisely to avoid the confusion which would otherwise result. Our approach is generally rigorous, but some of the harder or more technical proofs are either just sketched or omitted altogether. (However, a few harder proofs that are not available in that form elsewhere have been included, in particular those on sets with large intersection and on random fractals.) Suitable diagrams can be a help in understanding the proofs, many of which are of a geometric nature. Some diagrams are included in the book; the reader may find it helpful to draw others.

Chapter 1 begins with a rapid survey of some basic mathematical concepts and notation, for example, from the theory of sets and functions, which are used throughout the book. It also includes an introductory section on measure theory and mass distributions which, it is hoped, will be found adequate. The section on probability theory may be helpful for the chapters on random fractals and Brownian motion.

With the wide variety of topics covered, it is impossible to be entirely consistent in the use of notation, and inevitably, sometimes, there has to be a compromise between consistency within the book and standard usage.

In the past few years, fractals have become enormously popular as an art form, with the advent of computer graphics, and as a model of a wide variety of physical phenomena. Whilst it is possible in some ways to appreciate fractals with little or no knowledge of their mathematics, an understanding of the mathematics that can be applied to such a diversity of objects certainly enhances one's appreciation. The phrase 'the beauty of fractals' is often heard – it is the author's belief that much of their beauty is to be found in their mathematics.

It is a pleasure to acknowledge those who have assisted in the preparation of this book. Philip Drazin and Geoffrey Grimmett provided helpful comments on parts of the manuscript. Peter Shiarly gave valuable help with the computer drawings, and Aidan Foss produced some diagrams. I am indebted to Charlotte Farmer, Jackie Cowling and Stuart Gale of John Wiley & Sons for overseeing the production of the book.

Special thanks are due to David Marsh – not only did he make many useful comments on the manuscript and produce many of the computer pictures but he also typed the entire manuscript in a most expert way.

Finally, I would like to thank my wife Isobel for her support and encouragement, which extended to reading various drafts of the book.

April 1989

Kenneth J. Falconer
Bristol

Preface to the second edition

It is 13 years since *Fractal Geometry – Mathematical Foundations and Applications* was first published. In the meantime, the mathematics and applications of fractals have advanced enormously, with an ever-widening interest in the subject at all levels. The book was originally written for those working in mathematics and science who wished to know more about fractal mathematics. Over the past few years, with changing interests and approaches to mathematics teaching, many universities have introduced undergraduate and postgraduate courses on fractal geometry and a considerable number have been based on parts of this book.

Thus, this edition has two main aims. Firstly, it indicates some recent developments in the subject, with updated notes and suggestions for further reading. Secondly, more attention is given to the needs of students using the book as a course text, with extra details to help understanding, along with the inclusion of further exercises.

Parts of the book have been rewritten. In particular, multifractal theory has advanced considerably since the first edition was published, so the chapter on 'Multifractal Measures' has been completely rewritten. The notes and references have been updated. Numerous minor changes, corrections and additions have been incorporated, and some of the notation and terminology has been changed to conform with what has become standard usage. Many of the diagrams have been replaced to take advantage of the more sophisticated computer technology now available. Where possible, the numbering of sections, equations and figures has been left as in the first edition, so that earlier references to the book remain valid.

Further exercises have been added at the end of the chapters. Solutions to these exercises and additional supplementary material may be found on the World Wide Web at http://www.wileyeurope.com/fractal.

In 1997, a sequel, *Techniques in Fractal Geometry*, was published, presenting a variety of techniques and ideas current in fractal research. Readers wishing to study fractal mathematics beyond the bounds of this book may find the sequel helpful.

I am most grateful to all who have made constructive suggestions on the text. In particular, I am indebted to Carmen Fernández, Gwyneth Stallard and Alex Cain

for their help with this revision. I am also very grateful for the continuing support given to the book by the staff of John Wiley & Sons; and in particular, to Rob Calver and Lucy Bryan, for overseeing the production of this second edition and to John O'Connor and Louise Page for the cover design.

January 2003 **Kenneth J. Falconer**
St Andrews

Preface to the third edition

It is now 23 years since *Fractal Geometry – Mathematical Foundations and Applications* was first published and 10 years since the second edition. During those years, interest in the mathematics and applications of fractals has seen a phenomenal increase at all levels, with many mathematicians and scientists now involved in fractal-related topics. The book was originally written for researchers wanting to know more about fractals and their mathematics. However, many universities now present undergraduate and postgraduate courses on fractal geometry, often based on parts of this book, and the needs of students have been very much in mind during the revision. I am continually surprised by the number of researchers whom I meet who tell me that they first learnt about fractal geometry from this book.

This edition incorporates substantial changes from its predecessor with parts rewritten and new sections added. Student courses on fractal geometry usually present the simpler box-counting dimension before the more sophisticated Hausdorff dimension, so Chapters 2 and 3 have been reorganised in this way. The chapter on Brownian motion has been largely rewritten, and there are numerous minor changes and additions throughout the text. Some new sections have been added to give a glimpse of some of the recent ideas and directions, such as porosity and complex dimensions, that have evolved in fractal geometry.

When the first edition was written, the literature on fractals was comparatively limited and it was possible to include a reasonably comprehensive bibliography. In recent years, there has been an explosion in the number of research papers in the area and only a tiny proportion can be listed. Thus, the bibliography now focusses on papers that have a historical or innovative significance together with books and survey articles that provide overviews and many further references. The notes at the end of each chapter are a pointer to where next to go to find out more about a topic.

Some exercises at the end of the chapters have been modified and some more have been added, and, as before, solutions and other supplementary material may be found on the website http://www.wiley.com/go/fractal.

Those wishing to study fractals further may find helpful the sequel, *Techniques in Fractal Geometry* published in 1997, which is a natural continuation of this book and includes many ideas in use in current research.

Once again, I express my gratitude to the support given to the book by the staff of John Wiley & Sons and, in particular, to Richard Davies, Prachi Sinha-Sahay and Debbie Jupe for overseeing the production of this third edition. I am also very grateful to Ben Falconer for the new cover picture.

June 2013 **Kenneth J. Falconer**
St Andrews

Course suggestions

There is far too much material in this book for a standard length course on fractal geometry. Depending on the emphasis required, appropriate sections may be selected as a basis for an undergraduate or a postgraduate course.

A course for mathematics students could be based on the following sections.

(a) Mathematical background

1.1 Basic set theory;
1.2 Functions and limits;
1.3 Measures and mass distributions.

(b) Box-counting dimension

2.1 Box-counting dimensions;
2.2 Properties of box-counting dimensions.

(c) Hausdorff measures and dimension

3.1 Hausdorff measure;
3.2 Hausdorff dimension;
3.3 Calculation of Hausdorff dimension;
4.1 Basic methods of calculating dimensions.

(d) Iterated function systems

9.1 Iterated function systems;
9.2 Dimensions of self-similar sets;
9.3 Some variations;
10.2 Continued fraction examples.

(e) Graphs of functions

11.1 Dimensions of graphs, the Weierstrass function and self-affine graphs.

(f) Dynamical systems

13.1 Repellers and iterated function systems;
13.2 The logistic map.

(g) Iteration of complex functions

14.1 Sketch of general theory of Julia sets;
14.2 The Mandelbrot set;
14.3 Julia sets of quadratic functions.

Introduction

In the past, mathematics has been concerned largely with sets and functions to which the methods of classical calculus can be applied. Sets or functions that are not sufficiently smooth or regular have tended to be ignored as 'pathological' and not worthy of study. Certainly, they were regarded as individual curiosities and only rarely were thought of as a class to which a general theory might be applicable.

In recent years, this attitude has changed. It has been realised that a great deal can be said, and is worth saying, about the mathematics of non-smooth objects. Moreover, irregular sets provide a much better representation of many natural phenomena than do the figures of classical geometry. Fractal geometry provides a general framework for the study of such irregular sets.

We begin by looking briefly at a number of simple examples of fractals, and note some of their features.

The middle third Cantor set is one of the best known and most easily constructed fractals; nevertheless, it displays many typical fractal characteristics. It is constructed from a unit interval by a sequence of deletion operations (see Figure 0.1). Let E_0 be the interval [0, 1]. (Recall that $[a, b]$ denotes the set of real numbers x such that $a \leqslant x \leqslant b$.) Let E_1 be the set obtained by deleting the middle third of E_0, so that E_1 consists of the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. Deleting the middle thirds of these intervals gives E_2; thus, E_2 comprises the four intervals $\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right]$ and $\left[\frac{8}{9}, 1\right]$. We continue in this way, with E_k obtained by deleting the middle third of each interval in E_{k-1}. Thus, E_k consists of 2^k intervals each of length 3^{-k}. The *middle third Cantor set* F consists of the numbers that are in E_k for all k; mathematically, F is the intersection $\bigcap_{k=0}^{\infty} E_k$. The Cantor set F may be thought of as the limit of the sequence of sets E_k as k tends to infinity. It is obviously impossible to draw the set F itself, with its infinitesimal detail, so 'pictures of F' tend to be pictures of one of the E_k, which are a good approximation to F when k is reasonably large (see Figure 0.1).

At first glance, it might appear that we have removed so much of the interval [0, 1] during the construction of F, that nothing remains. In fact, F is an infinite (and indeed uncountable) set, which contains infinitely many numbers in every neighbourhood of each of its points. The middle third Cantor set F consists precisely of those numbers in [0, 1] whose base-3 expansion does not contain the digit 1,

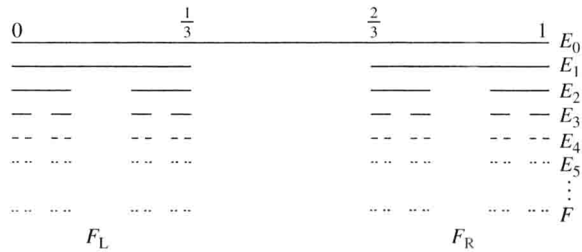

Figure 0.1 *Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note that* F_L *and* F_R*, the left and right parts of F, are copies of F scaled by a factor* $\frac{1}{3}$.

that is, all numbers $a_1 3^{-1} + a_2 3^{-2} + a_3 3^{-3} + \cdots$ with $a_i = 0$ or 2 for each i. To see this, note that to get E_1 from E_0, we remove those numbers with $a_1 = 1$; to get E_2 from E_1, we remove those numbers with $a_2 = 1$ and so on.

We list some of the features of the middle third Cantor set F; as we shall see, similar features are found in many fractals.

(i) F is self-similar. It is clear that the part of F in the interval $[0, \frac{1}{3}]$ and the part of F in $[\frac{2}{3}, 1]$ are both geometrically similar to F, scaled by a factor $\frac{1}{3}$. Again, the parts of F in each of the four intervals of E_2 are similar to F but scaled by a factor $\frac{1}{9}$, and so on. The Cantor set contains copies of itself at many different scales.

(ii) The set F has a 'fine structure'; that is, it contains detail at arbitrarily small scales. The more we enlarge the picture of the Cantor set, the more gaps become apparent to the eye.

(iii) Although F has an intricately detailed structure, the actual definition of F is very straightforward.

(iv) F is obtained by a recursive procedure. Our construction consisted of repeatedly removing the middle thirds of intervals. Successive steps give increasingly good approximations E_k to the set F.

(v) The geometry of F is not easily described in classical terms: neither is it the locus of the points that satisfy some simple geometric condition nor is it the set of solutions of any simple equation.

(vi) It is awkward to describe the local geometry of F – near each of its points are a large number of other points, separated by gaps of varying lengths.

(vii) Although F is in some ways quite a large set (it is uncountably infinite), its size is not quantified by the usual measures such as length – by any reasonable definition F has length zero.

Our second example, the von Koch curve, will also be familiar to many readers (see Figure 0.2). We let E_0 be a line segment of unit length. The set E_1 consists of

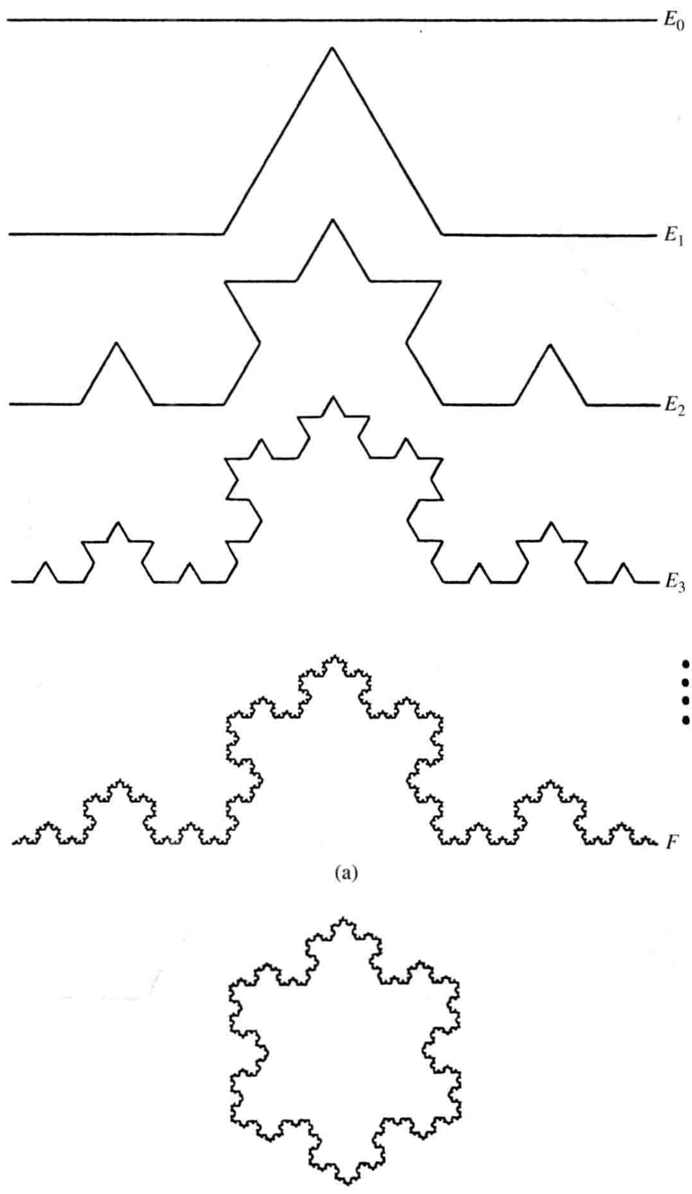

Figure 0.2 (a) Construction of the von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three von Koch curves fitted together to form a snowflake curve.

the four segments obtained by removing the middle third of E_0 and replacing it by the other two sides of the equilateral triangle based on the removed segment. We construct E_2 by applying the same procedure to each of the segments in E_1 and so on. Thus, E_k comes from replacing the middle third of each straight line segment of E_{k-1} by the other two sides of an equilateral triangle. When k is large, the curves E_{k-1} and E_k differ only in fine detail and as k tends to be infinity, the sequence of polygonal curves E_k approaches a limiting curve F, called the *von Koch curve*.

The von Koch curve has features in many ways similar to those listed for the middle third Cantor set. It is made up of four 'quarters' each similar to the whole, but scaled by a factor $\frac{1}{3}$. The fine structure is reflected in the irregularities at all scales; nevertheless, this intricate structure stems from a basically simple construction. Whilst it is reasonable to call F a curve, it is much too irregular to have tangents in the classical sense. A simple calculation shows that E_k is of length $\left(\frac{4}{3}\right)^k$; letting k tend to infinity implies that F has infinite length. On the other hand, F occupies zero area in the plane, so neither length nor area provides a very useful description of the size of F.

Many other sets may be constructed using such recursive procedures. For example, the *Sierpiński triangle* or *gasket* is obtained by repeatedly removing (inverted) equilateral triangles from an initial equilateral triangle of unit side length (see Figure 0.3). (For many purposes, it is better to think of this procedure as repeatedly replacing an equilateral triangle by three triangles of half the height.) A plane analogue of the Cantor set, a 'Cantor dust', is illustrated in Figure 0.4. At each stage, each remaining square is divided into 16 smaller squares of which

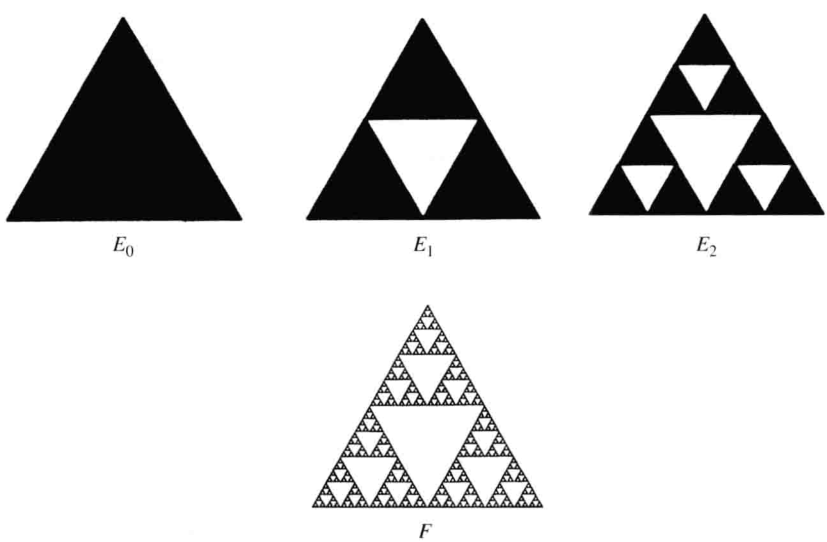

Figure 0.3 Construction of the Sierpiński triangle ($\dim_H F = \dim_B F = \log 3/\log 2$).