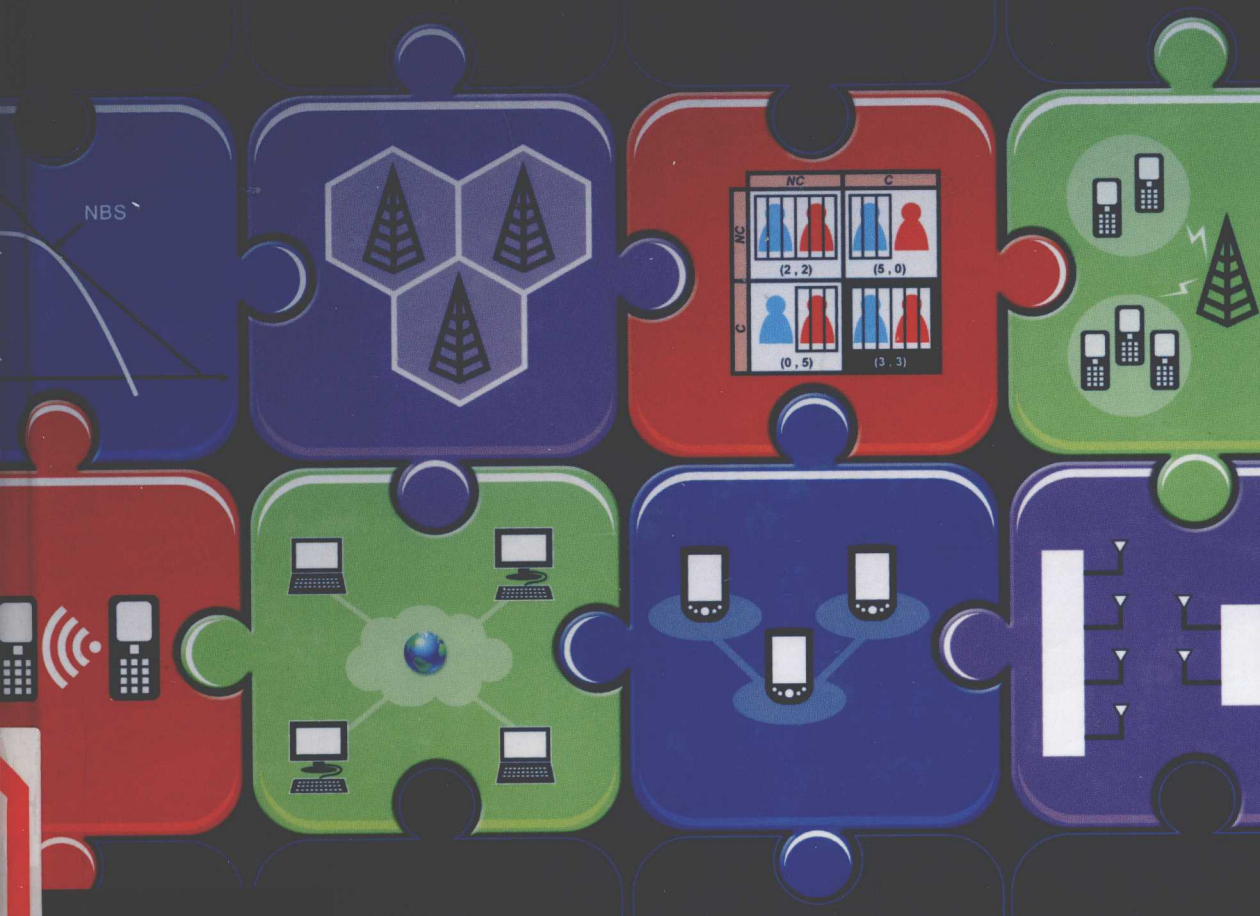


Game Theory

in Wireless and Communication Networks
Theory, Models, and Applications

**Zhu Han, Dusit Niyato, Walid Saad,
Tamer Başar, and Are Hjørungnes**





Game Theory for Wireless and Communication Networks

Theory, Models, and Applications

ZHU HAN

University of Houston

DUSIT NIYATO

Nanyang Technological University, Singapore

WALID SAAD

Princeton University

TAMER BAŞAR

University of Illinois at Urbana-Champaign

ARE HJØRUNGNES

University of Oslo, Norway



CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9780521196963

© Cambridge University Press 2012

This publication is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without the written
permission of Cambridge University Press.

First published 2012

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication Data

Game theory in wireless and communication networks : theory, models,
and applications / Zhu Han... [et al.].

p. cm.

Includes bibliographical references and index.

ISBN 978-0-521-19696-3 (hardback)

1. Wireless communication systems. 2. Mobile communication systems. 3. Computer
networks. 4. Telecommunication systems. 5. Game theory. I. Han, Zhu, 1974– II. Title.
TK5103.2.G35 2011

621.384015193–dc23 2011014906

ISBN 978-0-521-19696-3 Hardback

Cambridge University Press has no responsibility for the persistence or
accuracy of URLs for external or third-party internet websites referred to in
this publication, and does not guarantee that any content on such websites is,
or will remain, accurate or appropriate. Information regarding prices, travel
timetables, and other factual information given in this work is correct at
the time of first printing but Cambridge University Press does not guarantee
the accuracy of such information thereafter.

Game Theory in Wireless and Communication Networks

This unified treatment of game theory focuses on finding state-of-the-art solutions to issues surrounding the next generation of wireless and communication networks. Future networks will rely on autonomous and distributed architectures to improve the efficiency and flexibility of mobile applications, and game theory provides the ideal framework for designing efficient and robust distributed algorithms. This book enables readers to develop a solid understanding of game theory, its applications, and its use as an effective tool for addressing various problems in wireless communication and networking.

The key results and tools of game theory are covered, as are various real-world technologies including 3G/4G networks, wireless LANs, sensor networks, cognitive networks, and Internet networks. The book also covers a wide range of techniques for modeling, designing, and analyzing communication networks using game theory, as well as state-of-the-art distributed design techniques. This is an ideal resource for communications engineers, researchers, and graduate and undergraduate students.

Zhu Han is an Assistant Professor of Electrical and Computer Engineering at the University of Houston. He was awarded his Ph.D. in Electrical Engineering from the University of Maryland, College Park, in 2003 and worked for two years in industry as an R&D Engineer for JDSD.

Dusit Niyato is an Assistant Professor in the School of Computer Engineering at the Nanyang Technological University (NTU), Singapore. He received his Ph.D. in Electrical and Computer Engineering from the University of Manitoba, Canada, in 2008.

Walid Saad is an Assistant Professor at the Electrical and Computer Engineering Department at the University of Miami. His research interests include applications of game theory in wireless networks, small cell networks, cognitive radio, wireless communication systems (UMTS, WiMAX, LTE, etc), and smart grids.

Tamer Başar is a Swanlund Chair holder and CAS Professor of Electrical and Computer Engineering at the University of Illinois at Urbana-Champaign. He is a member of the US National Academy of Engineering, a Fellow of the IEEE and the IFAC, founding president of the ISDG, and current president of the AACC.

Are Hjørungnes was a Professor in the Faculty of Mathematics and Natural Sciences at the University of Oslo, Norway. He was a Senior Member of the IEEE and received his Ph.D. from the Norwegian University of Science and Technology in 2000.

While on a sabbatical at the University of Hawaii, our colleague and co-author, Dr. Are Hjørungnes, went missing and passed away during a mountain run on the island of Oahu. Words fail to express our sadness and sorrow in losing our dear friend. Are, you will remain forever engraved in our hearts and memories, as the Viking who was stronger than life itself. We will always remember your openness, great spirit, and technical brilliance. We would like to dedicate this book to you, as your efforts and perseverance were instrumental in the completion of this work.

May your soul rest in peace.

ZH, DN, WS, TB

To my daughter, Melody Han — Zhu Han

To my family — Dusit Niyato

To my wife Mary and my son Karim — Walid Saad

To my wife, Tangül — Tamer Başar

To my grandmother, Margit — Are Hjørungnes

Preface

With the recent advances in telecommunications technologies, wireless networking has become ubiquitous because of the great demand created by pervasive mobile applications. The convergence of computing, communications, and media will allow users to communicate with each other and access any content at any time and at any place. Future wireless networks are envisioned to support various services such as high-speed access, telecommuting, interactive media, video conferencing, real-time Internet games, e-business ecosystems, smart homes, automated highways, and disaster relief. Yet many technical challenges remain to be addressed in order to make this wireless vision a reality. A critical issue is devising *distributed* and *dynamic* algorithms for ensuring a robust network operation in time-varying and heterogeneous environments. Therefore, in order to support tomorrow's wireless services, it is essential to develop efficient mechanisms that provide an optimal cost-resource-performance tradeoff and that constitute the basis for next-generation ubiquitous and autonomic wireless networks.

Game theory is a formal framework with a set of mathematical tools to study the complex interactions among interdependent rational players. For more than half a century, game theory has led to revolutionary changes in economics, and it has found a number of important applications in politics, sociology, psychology, communication, control, computing, and transportation, to list only a few. During the past decade, there has been a surge in research activities that employ game theory to model and analyze modern communication systems. This is mainly due to (i) the emergence of the Internet as a global platform for computation and communication, which has sparked the development of large-scale, distributed, and heterogeneous communication systems; (ii) the deregulation of the telecommunications industry, and the dramatic improvement in computation power, which has made it possible for various network entities to make independent and selfish decisions; and (iii) the need for robust designs against uncertainties, e.g., in security situations that can sometimes be modeled as games of users with malicious intent.

Consequently, combining game theory with the design of efficient distributed algorithms for wireless networks is desirable but at the same time challenging. On the one hand, wireless network users are generally selfish in nature. For instance, distributed mobile users tend to maximize their own performance, regardless of how this maximization affects the other users in the network, subsequently giving rise to competitive scenarios. On the other hand, in some scenarios, cooperation is required among wireless network users for performance enhancement. These situations recently motivated researchers and engineers to adopt game-theoretic techniques for characterizing competition and

cooperation in wireless networks. As a result, game theory has been applied to solve many problems in wireless systems, e.g., those that arise in power control, network formation, admission control, cognitive radio, and traffic relaying. In fact, game theory provides solid mathematical tools for analyzing competition and cooperation in an ensemble of multiple players having individual self-interests. Various solution concepts from game theory are highly appropriate for communications and networking problems, such as equilibrium solutions that are desirable in competitive scenarios, since they lead to designs that are robust to the deviations made by any player. There are many popular wireless and communications applications that have recently explored game-theoretic techniques, including, but not limited to, cognitive radio, heterogeneous wireless networks, cellular networks, cooperative networks, and multi-hop networks. It is now commonly acknowledged that within the rich landscape of game theory, new aspects of network design (e.g., with cooperative and non-cooperative behaviors of the wireless entities) can be investigated using appropriate solution concepts.

Although game theory has been applied to wireless communications and networking for many years, there are only a few books that allow researchers, engineers, and graduate/undergraduate students to study game theory from an engineering perspective. On the one hand, most of the existing game theory books focus on the mathematical and economical aspects, which are considerably different from the engineering (and particularly the application-oriented) perspective. On the other hand, the wireless communications and networking books focus mainly on system optimization or control techniques while overlooking distributed algorithms. In addition, the cooperative and non-cooperative behaviors of the network entities (e.g., users or service providers) cannot be modeled and analyzed effectively using the techniques presented in these books. Therefore, there is a need to develop a comprehensive and useful reference source that can provide complete coverage on how to adequately apply game theory to the design of wireless communications and networking.

In this regard, this book not only focuses on the description of the main aspects of game theory in the context of wireless communications, but also provides an extensive review of the applications of game theory in wireless communications and networking problems. In a nutshell, it provides a comprehensive treatment of game theory in wireless communications and networking. The topics range from the basic concepts of game theory to the state of the art of analysis, design, and optimization of game-theoretic techniques for wireless and communication networks. The three main objectives of this book are as follows:

- This book introduces the basics of game theory from an engineering perspective. In particular, the basics of game theory are explained and discussed in the context of wireless communications and networking. For example, the book provides a clear description of the main game-theoretic entities in a communication environment (e.g., the players, their strategies, utilities and payoffs, and the physical meaning, in a wireless network environment, of the different game-theoretic concepts such as equilibria).
- This book provides an extensive review/survey of the applications of game theory to wireless communications and networking. With this review/survey of applications,

readers can understand how game theory can be applied in different wireless systems and can acquire an in-depth knowledge of the recent developments in this area. In this context, this book presents tutorial-like chapters that explain, clearly and concisely, how game-theoretic techniques can be applied to solving state-of-the-art wireless communications problems. In particular, the benefits of using game theory in wireless communications environments are emphasized. The target audience of this book are researchers, engineers, and undergraduate and graduate students who are looking for a self-contained book from which to learn game theory and its application to multi-player decision-making problems in wireless and other engineering systems.

- Most of the research in this field has been focused on applying standard game-theoretic models and techniques to several limited topics, such as power control in wireless networks and routing in wire-line networks. However, game theory is a very powerful tool and can help us better understand many other aspects of communication networks. The goals of this book are to provide the fundamental concepts of game theory and also to bring together the state-of-the-art research contributions that address the major opportunities and challenges of applying game theory in wireless engineering problems. The applications presented here are varied and cover a significant part of the most recent challenges and problems in wireless communications and networking systems. In this respect, we believe that this book will be useful to a variety of readers from the wireless communications and networking fields. The material from this book can be used to design and develop more efficient, scalable, and robust communication protocols.

To summarize, the key features of this book are

- a unified view of game-theoretic approaches to wireless networks
- comprehensive treatment of state-of-the-art distributed techniques for wireless communications problems
- coverage of a wide range of techniques for modeling, designing, and analyzing of wireless networks using game theory
- an outline of the key research issues related to wireless applications of game theory.

We would like to thank Dr. K. J. Ray Liu, Dr. Vincent Poor, Dr. John M. Cioffi, Dr. Luiz DaSilva, Dr. Allen MacKenzie, Dr. Mérouane Debbah, Dr. Ekram Hossain, Dr. Jianwei Huang, Dr. Ninoslav Marina, Dr. Guan-Ming Su, Dr. Yan Sun, Dr. Husheng Li, Dr. Beibei Wang, Dr. Charles Pandana, Dr. Zhu Ji, Dr. Rong Zheng, Dr. Xinbing Wang, Dr. Amir Leshem, Dr. Tansu Alpcan, Dr. Eduard Jorswieck, Mr. Quanyan Zhu, Dr. Eitan Altman, Dr. Corinne Touati, and Dr. María Ángeles Vázquez-Castro for their support on the book. We also would like to acknowledge the support of Mr. Ray Hardesty for text-editing and Ms. Jessy Stephan for her book cover design.

We would also like to acknowledge various granting agencies that supported part of the work reported in this book. These agencies are the US NSF through grants CNS-0905556, CNS-0910461, CNS-0953377, and ECCS-1028782; NTU Start-Up Grant – Project “Radio Resource Management in Heterogeneous Wireless Networks”; Singapore Ministry of Education (MOE) AcRF Tier 1 – Project “Radio Resource Management

over Cognitive Radio Networks”; A*STAR – SERC (Science and Engineering Research Council) “Data Value Chain as a Service” – Project “Design and Analysis of Cloud Computing for Data Value Chain: Operation Research Approach”; the Research Council of Norway for their funding of the VERDIKT Project “Mobile-to-Mobile Communication Systems (M2M)” (project number 183311/S10) and the FRITEK Project “Theoretical Foundations of Mobile Flexible Networks (THEFONE)” (project number 197565/V30); and the US AFOSR and DOE through grants AF FA9550-09-1-0249 and DOE SC0003879 ARRA.

Zhu Han
Dusit Niyato
Walid Saad
Tamer Başar
Are Hjørungnes

Contents

Preface

page xv

1	Introduction	1
1.1	Brief introduction to the history of game theory	1
1.2	Game theory in wireless and communication networks	3
1.3	Organization and targeted audience	4
1.3.1	Timeliness of the book	6
1.3.2	Outline of the book	9
2	Wireless networks: an introduction	14
2.1	Wireless channel models	15
2.1.1	Radio propagation	15
2.1.2	Interference channel	20
2.2	Categorization of wireless networks	21
2.2.1	3G cellular networks and beyond	21
2.2.2	WiMAX networks	25
2.2.3	WiFi networks	27
2.2.4	Wireless personal area networks	31
2.2.5	Wireless ad hoc networks	37
2.2.6	Wireless sensor networks	40
2.3	Advanced wireless technology	45
2.3.1	OFDM technology	45
2.3.2	Multiple-antenna systems	47
2.3.3	Cognitive radio	49

Part I Fundamentals of game theory

3	Non-cooperative games	55
3.1	Non-cooperative games: preliminaries	55
3.1.1	Introduction	55
3.1.2	Basics of non-cooperative games	56

3.2	Non-cooperative games in strategic form	58
3.2.1	Matrix games	58
3.2.2	Dominating strategies	61
3.2.3	Nash equilibrium	63
3.2.4	Static continuous-kernel games	65
3.2.5	Mixed strategies	69
3.2.6	Efficiency and equilibrium selection	72
3.3	Dynamic non-cooperative games	74
3.3.1	Non-cooperative games in extensive form	74
3.3.2	Repeated games	80
3.3.3	Stochastic games	84
3.4	Special classes of non-cooperative games	85
3.4.1	Potential games	85
3.4.2	Stackelberg games	88
3.4.3	Correlated equilibrium	91
3.4.4	Supermodular games	94
3.4.5	Wardrop equilibrium	96
3.5	Summary	100
4	Bayesian games	101
4.1	Overview of Bayesian games	101
4.1.1	Simple example	101
4.1.2	Static Bayesian game	102
4.1.3	Bayesian dynamic games in extensive form	104
4.1.4	Cournot duopoly model with incomplete information	105
4.1.5	Auction with incomplete information	107
4.2	Applications in wireless communications and networking	109
4.2.1	Packet-forwarding game	109
4.2.2	K -player Bayesian water-filling game	112
4.2.3	Channel-access game	116
4.2.4	Bandwidth-auction game	119
4.2.5	Bandwidth-allocation game	121
4.3	Summary	122
5	Differential games	124
5.1	Optimal-control theory	125
5.1.1	Dynamic programming	125
5.1.2	The maximum principle	126
5.2	Differential games	128
5.2.1	Main ingredients and general results	128
5.2.2	Stackelberg differential game	130
5.3	Applications of differential games in wireless communications and networking	136
5.4	Summary	137

6	Evolutionary games	138
6.1	The evolutionary process	139
6.1.1	Evolutionarily stable strategies	139
6.1.2	Replicator dynamics	141
6.1.3	The evolutionary game and reinforcement learning	143
6.2	Applications of evolutionary games in wireless communications and networking	144
6.2.1	Congestion control	144
6.2.2	Evolutionary game for the Aloha protocol	146
6.2.3	Evolutionary game for WCDMA access	148
6.2.4	Routing-potential game	149
6.2.5	Cooperative sensing in cognitive radio	151
6.2.6	TCP throughput adaptation	154
6.2.7	User churning behavior	158
6.2.8	Dynamic bandwidth allocation with evolutionary network selection	163
6.3	Summary	170
7	Cooperative games	171
7.1	Bargaining theory	171
7.1.1	Introduction	171
7.1.2	The Nash bargaining solution	172
7.1.3	Sample applications in wireless and communication networks	178
7.2	Coalitional game theory: basics	185
7.2.1	Introduction	185
7.2.2	Coalitional-game theory: preliminaries	185
7.3	Class I: canonical coalitional games	189
7.3.1	Main properties of canonical coalitional games	189
7.3.2	The core as a solution for canonical coalitional games	190
7.3.3	The Shapley value	195
7.3.4	The nucleolus	196
7.3.5	Sample applications in wireless and communication networks	198
7.4	Class II: coalition-formation games	203
7.4.1	Main properties of coalition-formation games	203
7.4.2	Impact of a coalitional structure on solution concepts for canonical coalitional games	203
7.4.3	Dynamic coalition-formation algorithms	205
7.4.4	Sample applications in wireless and communication networks	209
7.5	Class III: coalitional graph games	215
7.5.1	Main properties of coalitional graph games	215
7.5.2	Coalitional graph games and network-formation games	216
7.5.3	Sample applications in wireless and communication networks	219
7.6	Summary	220

8	Auction theory and mechanism design	221
8.1	Introduction and auction basics	222
8.2	Mechanism design	226
8.2.1	Equilibrium concepts	226
8.2.2	Participation and incentive compatibility	227
8.2.3	Revelation principle	228
8.2.4	Budget balance and efficiency	228
8.2.5	Groves mechanism	229
8.2.6	Impossibility and possibility	229
8.3	Special auctions	230
8.3.1	VCG auction	230
8.3.2	Share auction	232
8.3.3	Double auction	233
8.4	Examples of communication applications	235
8.4.1	Cognitive radio	236
8.4.2	Physical-layer security	248
8.5	Summary	251

Part II Applications of game theory in communications and networking

9	Cellular and broadband wireless access networks	255
9.1	Uplink power control in CDMA networks	257
9.1.1	Single-cell CDMA networks	258
9.1.2	Multi-cell wireless CDMA networks	263
9.2	Resource allocation in single-cell OFDMA networks	269
9.2.1	OFDMA resource-allocation model	270
9.2.2	Nash bargaining solution for subcarrier allocation	272
9.2.3	Algorithms for reaching the Nash bargaining solution	274
9.3	Power allocation in femtocell networks	279
9.3.1	Femtocell power control as a Stackelberg game	280
9.3.2	Multi-leader multi-follower Stackelberg equilibrium	284
9.3.3	Algorithm for reaching the Stackelberg equilibrium	286
9.4	IEEE 802.16 broadband wireless access networks	287
9.4.1	Resource allocation and admission control	287
9.4.2	Relay-station deployment in IEEE 802.16j	299
9.5	Network selection in multi-technology wireless networks	307
9.5.1	Network selection as a non-cooperative game	309
9.5.2	Network selection with incomplete information	311
9.6	Summary	320
10	Wireless local area networks	321
10.1	MAC protocol design	322
10.1.1	Static game	323

10.1.2	Dynamic game	324
10.1.3	Deviation detection and penalization	325
10.1.4	Related work	326
10.2	Random-access control	326
10.2.1	Choice of utility function	327
10.2.2	Dynamics of a random-access game	328
10.2.3	Extension with propagation delay and estimation error	329
10.2.4	Related work	329
10.3	Rate selection for VoIP service on WLAN	330
10.3.1	Game formulation	330
10.3.2	Payoff function	331
10.4	Access-point selection	332
10.4.1	Formulation of a population game	333
10.4.2	Price of anarchy	335
10.4.3	Access pricing	335
10.4.4	Related work	336
10.5	Admission control	337
10.5.1	Two-player game formulation	337
10.5.2	Interpretation of payoff	339
10.6	WiFi access-point pricing	339
10.6.1	Pricing scheme for direct payment	340
10.6.2	User with Web browsing	341
10.6.3	User with file transfer	342
10.6.4	Model for uncertain application	343
10.7	Summary	344

11**Multi-hop networks****345**

11.1	Routing-game basics	345
11.2	Cooperation enforcement and learning using a repeated game	349
11.2.1	System model and problem formulation	349
11.2.2	Self-learning cooperation-enforcing framework	350
11.2.3	Asynchronous network	352
11.2.4	Case analysis and performance evaluations	353
11.3	Hierarchical routing using a network-formation game	357
11.3.1	System model and game formulation	358
11.3.2	Hierarchical network-formation game solution	362
11.3.3	Hierarchical network-formation algorithm	364
11.3.4	Simulation results and analysis	366
11.4	Other typical approaches	369
11.4.1	Price-based solution	369
11.4.2	Truthfulness and security using auction theory	370
11.4.3	Evolutionary-game approach	372
11.5	Summary	373

12	Cooperative-transmission networks	375
12.1	Basics of cooperative transmission	376
12.1.1	Cooperative-transmission protocols	376
12.1.2	State of the art and impact on different layers	380
12.2	Non-cooperative game for relay selection and power control	380
12.2.1	Relay-selection and power-control problem	381
12.2.2	Stackelberg-game approach	382
12.3	Auction-theory-based resource allocation	389
12.3.1	Resource-allocation objectives	389
12.3.2	Share-auction approach	392
12.4	Cooperative transmission using a cooperative game in MANET	399
12.4.1	Selfishness in packet-forwarding networks	400
12.4.2	Cooperative transmission using a coalitional game	402
12.5	Cooperative routing	411
12.5.1	Cooperative-routing algorithms	412
12.5.2	WiMAX IEEE 802.16j	413
12.6	Summary	416
13	Cognitive-radio networks	418
13.1	Cooperative spectrum sensing	421
13.1.1	System model	421
13.1.2	Coalitional-game formulation	423
13.1.3	Centralized approach and performance comparison	426
13.2	Power allocation as a non-cooperative game	426
13.2.1	Underlay spectrum access and power allocation	426
13.2.2	Properties of the Nash equilibrium for power allocation	428
13.2.3	Distributed algorithm	429
13.2.4	Pigouvian taxation and social optimality	431
13.2.5	Related work	432
13.3	Medium access control	432
13.3.1	Channel allocation	433
13.3.2	Channel access	434
13.3.3	Distributed algorithms	435
13.4	Decentralized dynamic spectrum access	436
13.4.1	Overlay dynamic spectrum access	436
13.4.2	Utility function	438
13.4.3	Decentralized algorithm for channel access	439
13.4.4	Alternative algorithms	440
13.5	Radio resource competition based on a stochastic learning game	441
13.5.1	System model of radio resource competition	441
13.5.2	Auction mechanism	442
13.5.3	Secondary-user strategy	443
13.5.4	Learning algorithm	445

13.6	Cheat-proof strategies for open spectrum sharing	446
13.6.1	One-shot non-cooperative game	446
13.6.2	Cooperative strategy	447
13.6.3	Repeated games	448
13.6.4	Cheat-proof strategy	449
13.7	Spectrum leasing and cooperation	450
13.7.1	Game formulation with instantaneous CSI	451
13.7.2	Game formulation with long-term CSI	454
13.8	Service-provider competition for dynamic spectrum allocation	455
13.8.1	User demand	455
13.8.2	Optimal price	457
13.8.3	Related work	458
13.9	Summary	458
14	Internet networks	460
14.1	Combined flow control and routing in communication networks	462
14.1.1	Single user with multiple links	463
14.1.2	Multiple users with multiple parallel links	465
14.1.3	Sample Nash equilibria	471
14.2	Congestion control in networks with a single service provider	473
14.2.1	Pricing and congestion control	474
14.2.2	Non-cooperative Nash game between followers	476
14.2.3	Optimal pricing policy for the service provider	478
14.2.4	Network with a large number of followers	479
14.3	Pricing and revenue sharing for Internet service providers	481
14.3.1	Pricing game among Internet service providers	482
14.3.2	Revenue-sharing strategies	484
14.3.3	Distributed algorithm for finding a Nash equilibrium	485
14.4	Cooperative file sharing in peer-to-peer networks	487
14.4.1	Cooperative vs. non-cooperative file sharing	489
14.4.2	File sharing as a coalitional game in partition form	491
14.4.3	Distributed algorithm for coalition formation	493
14.4.4	Coalition formation in two-peer and N -peer networks	495
14.5	Summary	499
	<i>References</i>	501
	<i>Index</i>	530

1 Introduction

1.1 Brief introduction to the history of game theory

Game theory can be viewed as a branch of applied mathematics as well as of applied sciences. It has been used in the social sciences, most notably in economics, but has also penetrated into a variety of other disciplines such as political science, biology, computer science, philosophy, and, recently, wireless and communication networks. Even though game theory is a relatively young discipline, the ideas underlying it have appeared in various forms throughout history and in numerous sources, including the Bible, the Talmud, the works of Descartes and Sun Tzu, and the writings of Charles Darwin, and in the 1802 work *Considérations sur la Théorie Mathématique du Jeu* of André-Marie Ampère, who was influenced by the 1777 *Essai d'Arithmétique Morale* of Georges Louis Buffon. Nonetheless, the main basis of modern-day game theory can be considered an outgrowth of three seminal works:

- Augustin Cournot's *Mathematical Principles of the Theory of Wealth* in 1838, which gives an intuitive explanation of what would, over a century later, be formalized as the celebrated Nash equilibrium solution to non-cooperative games. Furthermore, Cournot's work provides an evolutionary or dynamic notion of the idea of a "best response," i.e., situations in which a player chooses the best action given the actions of other players, this being so for all players.
- Francis Ysidro Edgeworth's *Mathematical Physics* (1881), which demonstrated the notion of competitive equilibria in a two-person (as well as two-type) economy, and Emile Borel's *Algebre et Calcul des Probabilites* (*Comptes Rendus Academie des Sciences*, volume 184, 1927), which provided the first insight into mixed strategies, i.e., that randomization may support a stable outcome.
- While many other contributors hold places in the history of game theory, it is widely accepted that modern analysis started with John von Neumann and Oskar Morgenstern's 1944 book, *Theory of Games and Economic Behavior*, and was given its modern methodological framework by John Nash's seminal work on non-cooperative games and bargaining, which had von Neumann and Morgenstern's results as a first building block. It is worth mentioning that some two decades prior to this, in 1928, John von Neumann himself had resolved completely an open fundamental problem in zero-sum games, that *every finite two-player zero-sum game admits a saddle point in mixed strategies*, which is known as the *Minimax Theorem* [492]—a result which Emile Borel had conjectured to be false eight years earlier.