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Several Complex Variables

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Preface

The present book grew out of introductory lectures on the theory of functions of several variables. Its intent is to make the reader familiar, by the discussion of examples and special cases, with the most important branches and methods of this theory, among them, e.g., the problems of holomorphic continuation, the algebraic treatment of power series, sheaf and cohomology theory, and the real methods which stem from elliptic partial differential equations.

In the first chapter we begin with the definition of holomorphic functions of several variables, their representation by the Cauchy integral, and their power series expansion on Reinhardt domains. It turns out that, in contrast to the theory of a single variable, for $n \geqslant 2$ there exist domains $G, \hat{G} \subset \mathbb{C}^n$ with $G \subset \hat{G}$ and $G \neq \hat{G}$ such that each function holomorphic in G has a continuation on $\hat{G}$. Domains G for which such a $\hat{G}$ does not exist are called *domains of holomorphy*. In Chapter 2 we give several characterizations of these domains of holomorphy (theorem of Cartan–Thullen, Levi's problem). We finally construct the holomorphic hull $H(G)$ for each domain G, that is the largest (not necessarily schlicht) domain over $\mathbb{C}^n$ into which each function holomorphic on G can be continued.

The third chapter presents the Weierstrass formula and the Weierstrass preparation theorem with applications to the ring of convergent power series. It is shown that this ring is a factorization, a Noetherian, and a Hensel ring. Furthermore we indicate how the obtained algebraic theorems can be applied to the local investigation of analytic sets. One achieves deep results in this connection by using sheaf theory, the basic concepts of which are discussed in the fourth chapter. In Chapter V we introduce complex manifolds and give several examples. We also examine the different closures of $\mathbb{C}^n$ and the effects of modifications on complex manifolds.

Cohomology theory with values in analytic sheaves connects sheaf theory with the theory of functions on complex manifolds. It is treated and applied in Chapter VI in order to express the main results for domains of holomorphy and Stein manifolds (for example, the solvability of the Cousin problems).

The seventh chapter is entirely devoted to the analysis of real differentiability in complex notation, partial differentiation with respect to z, $\bar{z}$, and complex functional matrices, topics already mentioned in the first chapter. We define tangential vectors, differential forms, and the operators d, d', d''. The theorems of Dolbeault and de Rham yield the connection with cohomology theory.

The authors develop the theory in full detail and with the help of numerous figures. They refer to the literature for theorems whose proofs exceed the scope of the book. Presupposed are only a basic knowledge of differential and integral calculus and the theory of functions of one variable, as well as a few elements from vector analysis, algebra, and general topology. The book is written as an introduction and should be of interest to the specialist and the nonspecialist alike.

Göttingen, Spring 1976

H. Grauert
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Contents

CHAPTER I

Holomorphic Functions

Preliminaries

Let $\mathbb{C}$ be the field of complex numbers. If n is a natural number we call the set of ordered n-tuples of complex numbers the *n-dimensional complex number space*:

$$\mathbb{C}^n: = \{\mathfrak{z} = (z_1, \ldots, z_n) : z_\nu \in \mathbb{C} \text{ for } 1 \leqslant \nu \leqslant n\}.$$

Each component of a point $\mathfrak{z} \in \mathbb{C}^n$ can be decomposed uniquely into real and imaginary parts: $z_\nu = x_\nu + iy_\nu$. This gives a unique 1—1 correspondence between the elements $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ and the elements $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of $\mathbb{R}^{2n}$, the $2n$-dimensional space of real numbers.

$\mathbb{C}^n$ is a vector space: addition of two elements as well as the multiplication of an element of $\mathbb{C}^n$ by a (real or complex) scalar is defined componentwise. As a complex vector space $\mathbb{C}^n$ is n-dimensional; as a real vector space it is $2n$-dimensional. It is clear that the $\mathbb{R}$ vector space isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ leads to a topology on $\mathbb{C}^n$: For $\mathfrak{z} = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n$ let

$$\|\mathfrak{z}\|: = \left(\sum_{k=1}^{n} z_k \bar{z}_k\right)^{1/2} = \left(\sum_{k=1}^{n} (x_k^2 + y_k^2)\right)^{1/2},$$
$$\|\mathfrak{z}\|^*: = \max_{k=1,\ldots,n} (|x_k|, |y_k|).$$

Norms are defined on $\mathbb{C}^n$ by $\mathfrak{z} \mapsto \|\mathfrak{z}\|$ and $\mathfrak{z} \mapsto \|\mathfrak{z}\|^*$, with corresponding metrics given by

$$\operatorname{dist}(\mathfrak{z}_1, \mathfrak{z}_2): = \|\mathfrak{z}_1 - \mathfrak{z}_2\|,$$
$$\operatorname{dist}^*(\mathfrak{z}_1, \mathfrak{z}_2): = \|\mathfrak{z}_1 - \mathfrak{z}_2\|^*.$$

In each case we obtain a topology on $\mathbb{C}^n$ which agrees with the usual topology for $\mathbb{R}^{2n}$. Another metric on $\mathbb{C}^n$, defined by $|\mathfrak{z}| := \max_{k=1,\ldots,n} |z_k|$ and $\text{dist}'(\mathfrak{z}_1, \mathfrak{z}_2) := |\mathfrak{z}_1 - \mathfrak{z}_2|$, induces the usual topology too.

A *region* $B \subset \mathbb{C}^n$ is an open set (with the usual topology) and a *domain* an open, connected set. An open set $G \subset \mathbb{C}^n$ is called *connected* if one of the following two equivalent conditions is satisfied:

a. For every two points $\mathfrak{z}_1, \mathfrak{z}_2 \in G$ there is a continuous mapping $\varphi: [0, 1] \to \mathbb{C}^n$ with $\varphi(0) = \mathfrak{z}_1$, $\varphi(1) = \mathfrak{z}_2$, and $\varphi([0, 1]) \subset G$.
b. If $B_1, B_2 \subset G$ are open sets with $B_1 \cup B_2 = G$, $B_1 \cap B_2 = \varnothing$ and $B_1 \neq \varnothing$, then $B_2 = \varnothing$.

Definition. Let $B \subset \mathbb{C}^n$ be a region, $\mathfrak{z}_0 \in B$ a point. The set $C_B(\mathfrak{z}_0) := \{\mathfrak{z} \in B : \mathfrak{z}$ and $\mathfrak{z}_0$ can be joined by a path in $B\}$ is called the *component of* $\mathfrak{z}_0$ *in B.*

Remark. Let $B \subset \mathbb{C}^n$ be an open set. Then:

a. For each $\mathfrak{z} \in B$, $C_B(\mathfrak{z})$ and $B - C_B(\mathfrak{z})$ are open sets.
b. For each $\mathfrak{z} \in B$, $C_B(\mathfrak{z})$ is connected.
c. From $C_B(\mathfrak{z}_1) \cap C_B(\mathfrak{z}_2) \neq \varnothing$ it follows that $C_B(\mathfrak{z}_1) = C_B(\mathfrak{z}_2)$.
d. $B = \bigcup_{\mathfrak{z} \in B} C_B(\mathfrak{z})$
e. If G is a domain with $\mathfrak{z} \in G \subset B$, it follows that $G \subset C_B(\mathfrak{z})$.
f. B has at most countably many components.

The proof is trivial.

Finally for $\mathfrak{z}_0 \in \mathbb{C}^n$ we define:

$$U_\varepsilon(\mathfrak{z}_0) := \{\mathfrak{z} \in \mathbb{C}^n : \text{dist}(\mathfrak{z}, \mathfrak{z}_0) < \varepsilon\},$$
$$U_\varepsilon^*(\mathfrak{z}_0) := \{\mathfrak{z} \in \mathbb{C}^n : \text{dist}^*(\mathfrak{z}, \mathfrak{z}_0) < \varepsilon\},$$
$$U_\varepsilon'(\mathfrak{z}_0) := \{\mathfrak{z} \in \mathbb{C}^n : \text{dist}'(\mathfrak{z}, \mathfrak{z}_0) < \varepsilon\}.$$

1. Power Series

Let M be a subset of $\mathbb{C}^n$. A mapping f from M to $\mathbb{C}$ is called a complex function on M. The polynomials

$$p(\mathfrak{z}) = \sum_{\nu_1, \ldots, \nu_n = 0}^{m_1, \ldots, m_n} a_{\nu_1, \ldots, \nu_n} z_1^{\nu_1} \cdot \ldots \cdot z_n^{\nu_n}, \qquad a_{\nu_1, \ldots, \nu_n} \in \mathbb{C},$$

are particularly simple examples, defined on all of $\mathbb{C}^n$. In order to simplify notation we introduce multi-indices: let ν_i, $1 \leqslant i \leqslant n$, be non-negative integers and let $\mathfrak{z} = (z_1, \ldots, z_n)$ be a point of $\mathbb{C}^n$. Then we define:

$$\nu := (\nu_1, \ldots, \nu_n), \qquad |\nu| := \sum_{i=1}^{n} \nu_i, \qquad \mathfrak{z}^\nu := \prod_{i=1}^{n} z_i^{\nu_i}.$$

With this notation a polynomial has the form $p(\mathfrak{z}) = \sum_{\nu=0}^{m} a_\nu \mathfrak{z}^\nu$.

Def. 1.1. Let $\mathfrak{z}_0 \in \mathbb{C}^n$ be a point and for $|\nu| \geqslant 0$, a_ν be a complex number. Then the expression

$$\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$$

is called a *formal power series* about $\mathfrak{z}_0$.

Now such an expression has, as the name says, only a formal meaning. For a particular $\mathfrak{z}$ it does not necessarily represent a complex number. Since the multi-indices can be ordered in several ways it is not clear how the summation is to be performed. Therefore we must introduce a suitable notion of convergence.

Def. 1.2. Let $\mathfrak{I} := \{\nu = (\nu_1, \ldots, \nu_n) : \nu_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n\}$, and $\mathfrak{z}_1 \in \mathbb{C}^n$ be fixed. We say that $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z}_1 - \mathfrak{z}_0)^\nu$ *converges to the complex number* c if for each $\varepsilon > 0$ there exists a finite set $I_0 \subset \mathfrak{I}$ such that for any finite set I with $I_0 \subset I \subset \mathfrak{I}$

$$\left| \sum_{\nu \in I} a_\nu(\mathfrak{z}_1 - \mathfrak{z}_0)^\nu - c \right| < \varepsilon.$$

One then writes $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu = c$.

Convergence in this sense is synonymous with absolute convergence.

Def. 1.3. Let M be a subset of $\mathbb{C}^n$, $\mathfrak{z}_0 \in M$, f a complex function on M. One says that the power series $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ *converges uniformly* on M to $f(\mathfrak{z})$ if for each $\varepsilon > 0$ there is a finite set $I_0 \subset \mathfrak{I}$ such that

$$\left| \sum_{\nu \in I} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu - f(\mathfrak{z}) \right| < \varepsilon$$

for each finite I with $I_0 \subset I \subset \mathfrak{I}$ and each $\mathfrak{z} \in M$.

$\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ converges uniformly in the interior of a region B if the series converges uniformly in each compact subset of B.

Def. 1.4. Let $B \subset \mathbb{C}^n$ be a region and f be a complex function on B. f is called *holomorphic* in B if for each $\mathfrak{z}_0 \in B$ there is a neighborhood $U = U(\mathfrak{z}_0)$ in B and a power series $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ which converges on U to $f(\mathfrak{z})$.

Note that uniform convergence on U is not required. We show now why pointwise convergence suffices.

Def. 1.5. The point set $V = \{\mathfrak{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n : r_\nu \geqslant 0 \text{ for } 1 \leqslant \nu \leqslant n\}$ will be called *absolute space.* $\tau : \mathbb{C}^n \to V$ with $\tau(\mathfrak{z}) := (|z_1|, \ldots, |z_n|)$ is the *natural projection* of $\mathbb{C}^n$ onto V.

V is a subset of $\mathbb{R}^n$ and as such inherits the topology induced from $\mathbb{R}^n$ to V (relative topology). Then $\tau : \mathbb{C}^n \to V$ is a continuous surjective mapping. If $B \subset V$ is open, then $\tau^{-1}(B) \subset \mathbb{C}^n$ is also open.

Def. 1.6. Let $\mathfrak{r} \in V_+ := \{\mathfrak{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n : r_k > 0\}$, $\mathfrak{z}_0 \in \mathbb{C}^n$. Then $P_\mathfrak{r}(\mathfrak{z}_0) := \{\mathfrak{z} \in \mathbb{C}^n : |z_k - z_k^{(0)}| < r_k \text{ for } 1 \leqslant k \leqslant n\}$ is called the *polycylinder about* $\mathfrak{z}_0$ with *(poly-)radius* $\mathfrak{r}$. $T = T(P) := \{\mathfrak{z} \in \mathbb{C}^n : |z_k - z_k^{(0)}| = r_k\}$ is called the *distinguished boundary of P* (see Fig. 1).

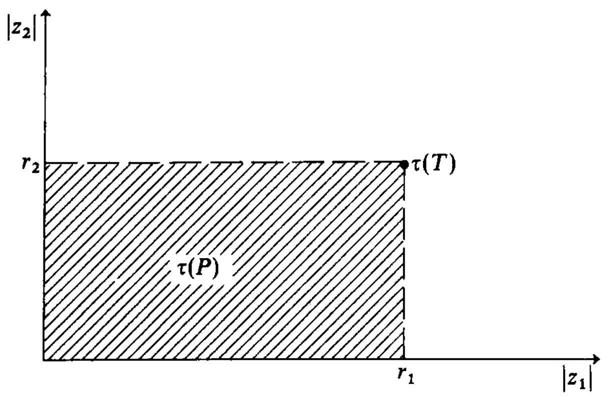

Figure 1. The image of a polycylinder in absolute space.

$P = P_\mathfrak{r}(\mathfrak{z}_0)$ is a convex domain in $\mathbb{C}^n$, and its distinguished boundary is a subset of the topological boundary ∂P of P. For $n = 2$ and $\mathfrak{z}_0 = 0$ the situation is easily illustrated: V is then a quadrant in $\mathbb{R}^2$, $\tau(P)$ is an open rectangle, and $\tau(T)$ is a point on the boundary of $\tau(P)$. Therefore

$$\begin{aligned} T &= \{\mathfrak{z} \in \mathbb{C}^2 : |z_1| = r_1, |z_2| = r_2\} \\ &= \{\mathfrak{z} = (r_1 \cdot e^{i\theta_1}, r_2 \cdot e^{i\theta_2}) \in \mathbb{C}^2 : 0 \leqslant \theta_1 < 2\pi, 0 \leqslant \theta_2 < 2\pi\} \end{aligned}$$

is a 2-dimensional torus. Similarly in the n-dimensional case we get an n-dimensional torus (the cartesian product of n circles).

If $\mathfrak{z}_1 \in \mathring{\mathbb{C}}^n := \{\mathfrak{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_k \neq 0 \text{ for } 1 \leqslant k \leqslant n\}$, then $P_{\mathfrak{z}_1} := \{\mathfrak{z} \in \mathbb{C}^n : |z_k| < |z_k^{(1)}| = r_k \text{ for } 1 \leqslant k \leqslant n\}$ is a polycylinder about 0 with radius $\mathfrak{r} = (r_1, \ldots, r_n)$.

Theorem 1.1. *Let* $\mathfrak{z}_1 \in \mathring{\mathbb{C}}^n$. *If the power series* $\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ *converges at* $\mathfrak{z}_1$, *then it converges uniformly in the interior of the polycylinder* $P_{\mathfrak{z}_1}$.

PROOF

1. Since the series converges at $\mathfrak{z}_1$, the set $\{a_\nu \mathfrak{z}_1^\nu : |\nu| \geqslant 0\}$ is bounded. Let $M \in \mathbb{R}$ be chosen so that $|a_\nu \mathfrak{z}_1^\nu| < M$ for all ν. If $\mathfrak{z}_1 \in \mathring{\mathbb{C}}^n$ and $0 < q < 1$ then $q \cdot \mathfrak{z}_1 \in \mathring{\mathbb{C}}^n$. Let $P^* : = P_{q \cdot \mathfrak{z}_1}$. For $\mathfrak{z} \in P^*$, $|\mathfrak{z}^\nu| = |z_1|^{\nu_1} \cdot \ldots \cdot |z_n|^{\nu_n} < |q \cdot z_1^{(1)}|^{\nu_1} \cdot \ldots \cdot |q \cdot z_n^{(1)}|^{\nu_n} = q^{\nu_1 + \cdots + \nu_n} \cdot |z_1^{(1)}|^{\nu_1} \cdot \ldots \cdot |z_n^{(1)}|^{\nu_n} = q^{|\nu|} \cdot |\mathfrak{z}_1^\nu|$, that is, $\sum_{\nu=0}^{\infty} |a_\nu| \cdot |\mathfrak{z}_1^\nu| \cdot q^{|\nu|}$ is a majorant of $\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ and therefore

$$M \cdot \sum_{\nu=0}^{\infty} q^{\nu_1 + \cdots + \nu_n} = M \cdot \left(\sum_{\nu_1=0}^{\infty} q^{\nu_1} \right) \cdot \ldots \cdot \left(\sum_{\nu_n=0}^{\infty} q^{\nu_n} \right) = M \cdot \left(\frac{1}{1-q} \right)^n.$$

The set $\mathfrak{I}$ of multi-indices is countable, so there exists a bijection $\Phi : \mathbb{N}_0 \to \mathfrak{I}$. Let $b_n(\mathfrak{z}) : = a_{\Phi(n)} \cdot \mathfrak{z}^{\Phi(n)}$. Then $\sum_{n=0}^{\infty} b_n(\mathfrak{z})$ is absolutely and uniformly convergent on P^*. Given $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} |b_n(\mathfrak{z})| < \varepsilon$ on P^*. Let $I_0 : = \Phi(\{0, 1, 2, \ldots, n_0\})$. If I is a finite set with $I_0 \subset I \subset \mathfrak{I}$, then $\{0, 1, \ldots, n_0\} \subset \Phi^{-1}(I)$, so

$$\begin{aligned} \left| \sum_{n=0}^{\infty} b_n(\mathfrak{z}) - \sum_{\nu \in I} a_\nu \mathfrak{z}^\nu \right| &= \left| \sum_{n=0}^{\infty} b_n(\mathfrak{z}) - \sum_{n \in \Phi^{-1}(I)} b_n(\mathfrak{z}) \right| \\ &= \left| \sum_{n \notin \Phi^{-1}(I)} b_n(\mathfrak{z}) \right| \leqslant \sum_{n=n_0+1}^{\infty} |b_n(\mathfrak{z})| < \varepsilon \text{ for } \mathfrak{z} \in P^*. \end{aligned}$$

But then $\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ is uniformly convergent in P^*.

2. Let $K \subset P_{\mathfrak{z}_1}$ be compact. $\{P_{q \cdot \mathfrak{z}_1} : 0 < q < 1\}$ is an open covering of $P_{\mathfrak{z}_1}$, and thus of K. But then there is a finite subcovering $\{P_{q_1 \cdot \mathfrak{z}_1}, \ldots, P_{q_\ell \cdot \mathfrak{z}_1}\}$. If we set $q : = \max(q_1, \ldots, q_\ell)$, then $K \subset P_{q \cdot \mathfrak{z}_1}$, and $P_{q \cdot \mathfrak{z}_1}$ is a P^* such as in 1). Therefore $\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ is uniformly convergent on K, which was to be shown. □

Next we shall examine on what sets power series converge. In order to be brief we choose $\mathfrak{z}_0 = 0$ as our point of expansion. The corresponding statements always hold in the general case.

Def. 1.7. An open set $B \subset \mathbb{C}^n$ is called a *Reinhardt domain* if $\mathfrak{z}_1 \in B \Rightarrow T_{\mathfrak{z}_1} : = \tau^{-1}\tau(\mathfrak{z}_1) \subset B$.

Comments. $T_{\mathfrak{z}_1}$ is the torus $\{\mathfrak{z} \in \mathbb{C}^n : |z_k| = |z_k^{(1)}|\}$. The conditions of definition 1.7 mean that $\tau^{-1}\tau(B) = B$; a Reinhardt domain is characterized by its image $\tau(B)$ in absolute space.

Theorem 1.2. *An open set $B \subset \mathbb{C}^n$ is a Reinhardt domain if and only if there exists an open set $W \subset V$ with $B = \tau^{-1}(W)$.*

PROOF

1. Let $B = \tau^{-1}(W)$, $W \subset V$ open. For $\mathfrak{z} \in B$, $\tau(\mathfrak{z}) \in W$; therefore $\tau^{-1}\tau(\mathfrak{z}) \subset \tau^{-1}(W) = B$.

2. Let B be a Reinhardt domain. Then $B = \tau^{-1}\tau(B)$ and it suffices to show that $\tau(B)$ is open in V. Assume that $\tau(B)$ is not open. Then there is a point $\mathfrak{r}_0 \in \tau(B)$ which is not an interior point of $\tau(B)$ and therefore is a cluster point of $V - \tau(B)$. Let $(\mathfrak{r}_j)$ be a sequence in $V - \tau(B)$ which converges to $\mathfrak{r}_0$. There are points $\mathfrak{z}_j \in \mathbb{C}^n$ with $\mathfrak{r}_j = \tau(\mathfrak{z}_j)$, so that $|z_p^{(j)}| = r_p^{(j)}$ for all j and $1 \leqslant p \leqslant n$. Since $(\mathfrak{r}_j)$ is convergent there is an $M \in \mathbb{R}$ such that $|r_p^{(j)}| < M$ for all j and p. Hence the sequence $(\mathfrak{z}_j)$ is also bounded. It must have a cluster point $\mathfrak{z}_0$, and a subsequence $(\mathfrak{z}_{j_\nu})$ with $\lim_{\nu \to \infty} \mathfrak{z}_{j_\nu} = \mathfrak{z}_0$. Since τ is continuous $\tau(\mathfrak{z}_0) = \lim_{\nu \to \infty} \tau(\mathfrak{z}_{j_\nu}) = \lim_{\nu \to \infty} \mathfrak{r}_{j_\nu} = \mathfrak{r}_0$. B is a Reinhardt domain; it follows that $\mathfrak{z}_0 \in \tau^{-1}(\mathfrak{r}_0) \subset \tau^{-1}\tau(B) = B$. B is an open neighborhood of $\mathfrak{z}_0$; therefore almost all $\mathfrak{z}_{j_\nu}$ must lie in B, and then almost all $\mathfrak{r}_{j_\nu} = \tau(\mathfrak{z}_{j_\nu})$ must lie in $\tau(B)$. This is a contradiction, and therefore $\tau(B)$ is open. □

The image of a Reinhardt domain in absolute space is always an open set (of arbitrary form), and the inverse image of this set is again the domain.

Def. 1.8. Let $G \subset \mathbb{C}^n$ be a Reinhardt domain.

1. G is called *proper* if

   a. G is connected, and
   b. $0 \in G$.

2. G is called *complete* if

$$\mathfrak{z}_1 \in G \cap \mathring{\mathbb{C}}^n \Rightarrow P_{\mathfrak{z}_1} \subset G.$$

Figure 2 illustrates Def. 1.8. for the case $n = 2$ in absolute space.

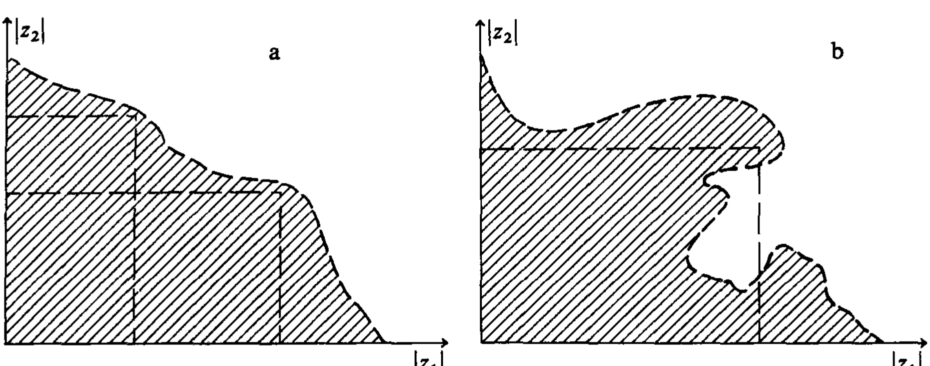

Figure 2. (a) Complete Reinhardt domain; (b) Proper Reinhardt domain.

For $n = 1$ Reinhardt domains are the unions of open annuli. There is no difference between complete and proper Reinhardt domains in this case; we are dealing with open circular discs.

Clearly for $n > 1$ the polycylinders and balls $K = \{\mathfrak{z}: |z_1|^2 + \cdots + |z_n|^2 < R^2\}$ are proper and complete Reinhardt domains. In general:

Theorem 1.3. *Every complete Reinhardt domain is proper.*

PROOF. Let G be a complete Reinhardt domain. There exists a point $\mathfrak{z}_1 \in G \cap \mathring{\mathbb{C}}^n$, and by definition $0 \in P_{\mathfrak{z}_1} \subset G$. It remains to show that G is connected.

a. Let $\mathfrak{z}_1 \in G$ be a point in a general position (i.e., $\mathfrak{z}_1 \in G \cap \mathring{\mathbb{C}}^n$). Then the connecting line segment between $\mathfrak{z}_1$ and 0 lies entirely within $P_{\mathfrak{z}_1}$ and hence within G.

b. $\mathfrak{z}_1$ lies on one of the "axes." Since G is open there exists a neighborhood $U_\varepsilon(\mathfrak{z}_1) \subset G$, and we can find a point $\mathfrak{z}_2 \in U_\varepsilon(\mathfrak{z}_1) \cap \mathring{\mathbb{C}}^n$. Hence there is a path in U_ε which connects $\mathfrak{z}_1$ and $\mathfrak{z}_2$, and a path in G which connects $\mathfrak{z}_2$ and 0. Together they give a path in G which joins $\mathfrak{z}_1$ and 0.

From (a) and (b) it follows that G is connected. □

Let $\mathfrak{P}(\mathfrak{z}) = \sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ be a power series about zero. The set $M \subset \mathbb{C}^n$ on which $\mathfrak{P}(\mathfrak{z})$ converges is called the *convergence set* of $\mathfrak{P}(\mathfrak{z})$. $\mathfrak{P}(\mathfrak{z})$ always converges in $\mathring{M}$ and diverges outside $\bar{M}$. $B(\mathfrak{P}(\mathfrak{z})) := \mathring{M}$ is called the *region of convergence* of the power series $\mathfrak{P}(\mathfrak{z})$.

Theorem 1.4. *Let $\mathfrak{P}(\mathfrak{z}) = \sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu$ be a formal power series in $\mathbb{C}^n$. Then the region of convergence $B = B(\mathfrak{P}(\mathfrak{z}))$ is a complete Reinhardt domain. $\mathfrak{P}(\mathfrak{z})$ converges uniformly in the interior of B.*

PROOF

1. Let $\mathfrak{z}_1 \in B$. Then $U'_\varepsilon(\mathfrak{z}_1) = \{\mathfrak{z} \in \mathbb{C}^n: |\mathfrak{z} - \mathfrak{z}_1| < \varepsilon\} = U_\varepsilon(z_1^{(1)}) \times \cdots \times U_\varepsilon(z_n^{(1)})$ is a polycylinder about $\mathfrak{z}_1$ with radius $(\varepsilon, \ldots, \varepsilon)$. For a sufficiently small ε, $U'_\varepsilon(\mathfrak{z}_1)$ lies in B. For $k = 1, \ldots, n$ we can find a $z_k^{(2)} \in U_\varepsilon(z_k^{(1)})$ such that $|z_k^{(2)}| > |z_k^{(1)}|$. Let $\mathfrak{z}_2 := (z_1^{(2)}, \ldots, z_n^{(2)})$. Then $\mathfrak{z}_2 \in B$ and $\mathfrak{z}_1 \in P_{\mathfrak{z}_2}$. For each point $\mathfrak{z}_1 \in B$ choose such a fixed point $\mathfrak{z}_2$.

2. If $\mathfrak{z}_1 \in B$, then there is a $\mathfrak{z}_2 \in B$ with $\mathfrak{z}_1 \in P_{\mathfrak{z}_2}$. $\mathfrak{P}(\mathfrak{z})$ converges at $\mathfrak{z}_2$, therefore in $P_{\mathfrak{z}_2}$ (from Theorem 1.1). Hence $P_{\mathfrak{z}_2} \subset B$. Since $P_{\mathfrak{z}_1} \subset P_{\mathfrak{z}_2}$ and $T_{\mathfrak{z}_1} \subset P_{\mathfrak{z}_2}$, it follows that B is a complete Reinhardt domain.

3. Let $P^*_{\mathfrak{z}_1} := P_{\mathfrak{z}_2}$ where $\mathfrak{z}_2$ is chosen for $\mathfrak{z}_1$ as in 1). Clearly $B = \bigcup_{\mathfrak{z}_1 \in B} P^*_{\mathfrak{z}_1}$. Now for each $\mathfrak{z}_2$ select a q with $0 < q < 1$ and such that $\mathfrak{z}_3 := (1/q)\mathfrak{z}_2$ lies in B. This is possible and it follows that for each $\mathfrak{z}_1 \in B$ $\mathfrak{P}(\mathfrak{z})$ is uniformly convergent in $P^*_{\mathfrak{z}_1}$. If $K \subset B$ is compact, then K can be covered by a finite number of sets $P^*_{\mathfrak{z}_1}$. Therefore $\mathfrak{P}(\mathfrak{z})$ converges uniformly on K. □

The question arises whether every complete Reinhardt domain is the region of convergence for some power series. This is not true; additional properties are necessary. However, we shall not pursue this matter here.

Since each complete Reinhardt domain is connected, we can speak of the *domain of convergence* of a power series. We now return to the notion of holomorphy.

Let f be a holomorphic function on a region B, $\mathfrak{z}_0$ a point in B. Let the power series $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ converge to $f(\mathfrak{z})$ in a neighborhood U of $\mathfrak{z}_0$. Then there is a $\mathfrak{z}_1 \in U$ with $z_\nu^{(1)} \neq z_\nu^{(0)}$ for $1 \leqslant \nu \leqslant n$ and $P_{\mathfrak{r}(\mathfrak{z}_1 - \mathfrak{z}_0)}(\mathfrak{z}_0) \subset U$. Now let $0 < \varepsilon < \min_{\nu=1,\dots,n} (|z_\nu^{(1)} - z_\nu^{(0)}|)$. From Theorem 1.1 the series converges uniformly on $U'_\varepsilon(\mathfrak{z}_0)$. For each $\nu \in \mathfrak{I}$ one can regard $a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ as a complex-valued function on $\mathbb{R}^{2n}$. This function is clearly continuous at $\mathfrak{z}_0$ and consequently the limit function is continuous at $\mathfrak{z}_0$. We have:

Theorem 1.5. *Let $B \subset \mathbb{C}^n$ be a region, and f a function holomorphic on B. Then f is continuous on B.*

2. Complex Differentiable Functions

Def. 2.1. Let $B \subset \mathbb{C}^n$ be a region, $f: B \to \mathbb{C}$ a complex function. f is called *complex differentiable at* $\mathfrak{z}_0 \in B$ if there exist complex functions $\Delta_1, \dots, \Delta_n$ on B which are all continuous at $\mathfrak{z}_0$ and which satisfy the equality $f(\mathfrak{z}) = f(\mathfrak{z}_0) + \sum_{\nu=1}^{n} (z_\nu - z_\nu^{(0)})\, \Delta_\nu(\mathfrak{z})$ in B.

Differentiability is a local property. If there exists a neighborhood $U = U(\mathfrak{z}_0) \subset B$ such that $f|U$ is complex differentiable at $\mathfrak{z}_0$, then $f|B$ is complex differentiable at $\mathfrak{z}_0$ since the functions $\Delta_\nu(\mathfrak{z})$ can be continued outside U in such a way that the desired equation holds.

At $\mathfrak{z}_0$ the following is true:

Theorem 2.1. *Let $B \subset \mathbb{C}^n$ be a region and $f: B \to \mathbb{C}$ complex differentiable at $\mathfrak{z}_0 \in B$. Then the values of the functions $\Delta_1, \dots, \Delta_n$ at $\mathfrak{z}_0$ are uniquely determined.*

PROOF. $E_\nu := \{\mathfrak{z} \in \mathbb{C}^n : z_\lambda = z_\lambda^{(0)} \text{ for } \lambda \neq \nu\}$ is a complex one-dimensional plane. Let $B_\nu := \{\zeta \in \mathbb{C} : (z_1^{(0)}, \dots, z_{\nu-1}^{(0)}, \zeta, z_{\nu+1}^{(0)}, \dots, z_n^{(0)}) \in E_\nu \cap B\}$. $f_\nu^*(z_\nu) := f(z_1^{(0)}, \dots, z_{\nu-1}^{(0)}, z_\nu, z_{\nu+1}^{(0)}, \dots, z_n^{(0)})$ defines a complex function on B_ν. Since f is differentiable at $\mathfrak{z}_0$, we have on B_ν

$$\begin{aligned} f_\nu^*(z_\nu) &= f(z_1^{(0)}, \dots, z_{\nu-1}^{(0)}, z_\nu, z_{\nu+1}^{(0)}, \dots, z_n^{(0)}) \\ &= f(\mathfrak{z}_0) + (z_\nu - z_\nu^{(0)}) \cdot \Delta_\nu(z_1^{(0)}, \dots, z_\nu, \dots, z_n^{(0)}) \\ &= f_\nu^*(z_\nu^{(0)}) + (z_\nu - z_\nu^{(0)}) \cdot \Delta_\nu^*(z_\nu). \end{aligned}$$

Thus $\Delta_\nu^*(z_\nu) := \Delta_\nu(z_1^{(0)}, \ldots, z_{\nu-1}^{(0)}, z_\nu, z_{\nu+1}^{(0)}, \ldots, z_n^{(0)})$ is continuous at $z_\nu^{(0)}$. Therefore $f_\nu^*(z_\nu)$ is complex differentiable at $z_\nu^{(0)} \in \mathbb{C}^n$, and $\Delta_\nu^*(z_\nu^{(0)}) = \Delta_\nu(\mathfrak{z}_0)$ is uniquely determined. This holds for each ν. □

Def. 2.2. Let the complex function f defined on the region $B \subset \mathbb{C}^n$ be complex differentiable at $\mathfrak{z}_0 \in B$. If $f(\mathfrak{z}) = f(\mathfrak{z}_0) + \sum_{\nu=1}^{n} (z_\nu - z_\nu^{(0)})\, \Delta_\nu(\mathfrak{z})$, then we call $\Delta_\nu(\mathfrak{z}_0)$ the *partial derivative of f with respect to z_ν at $\mathfrak{z}_0$*, and write $\Delta_\nu(\mathfrak{z}_0) = \dfrac{\partial f}{\partial z_\nu}(\mathfrak{z}_0) = f_{z_\nu}(\mathfrak{z}_0) = f_{,\nu}(\mathfrak{z}_0)$.

Theorem 2.2. *Let $B \subset \mathbb{C}^n$ be a region and f complex differentiable at $\mathfrak{z}_0 \in B$. Then f is continuous at $\mathfrak{z}_0$.*

Proof. We have $f(\mathfrak{z}) = f(\mathfrak{z}_0) + \sum_{\nu=1}^{n} (z_\nu - z_\nu^{(0)})\, \Delta_\nu(\mathfrak{z})$; the right side of this equation is clearly continuous at $\mathfrak{z}_0$. □

Let $B \subset \mathbb{C}^n$ be a region. f is called complex differentiable on B if f is complex differentiable at each point of B.

Sums, products, and quotients (with nonvanishing denominators) of complex differentiable functions are again complex differentiable. The proof is analogous to the real case, and we do not present it here.

Theorem 2.3. *Let $B \subset \mathbb{C}^n$ be a region, f holomorphic in B. Then f is complex differentiable in B.*

Proof. Let $\mathfrak{z}_0 \in B$. Then there is a neighborhood $U = U(\mathfrak{z}_0)$ and a power series $\sum_{\nu=0}^{\infty} a_\nu(\mathfrak{z} - \mathfrak{z}_0)^\nu$ which in U converges uniformly to $f(\mathfrak{z})$. Without loss of generality let $\mathfrak{z}_0 = 0$. Then

$$\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}^\nu = a_{0\cdots 0} + z_1 \cdot \sum_{\nu_1 \geq 1} a_{\nu_1 \cdots \nu_n} z_1^{\nu_1 - 1} \cdot z_2^{\nu_2} \cdots z_n^{\nu_n}$$
$$+ z_2 \cdot \sum_{\nu_2 \geq 1} a_{0, \nu_2 \cdots \nu_n} z_2^{\nu_2 - 1} \cdot z_3^{\nu_3} \cdots z_n^{\nu_n} + \cdots + z_n \cdot \sum_{\nu_n \geq 1} a_{0 \cdots 0, \nu_n} z_n^{\nu_n - 1}.$$

For now, this decomposition has only formal meaning. Choose a polycylinder of the form $P = U_\varepsilon(0) \times \cdots \times U_\varepsilon(0) \subset U(0)$ and a point $\mathfrak{z}_1 \in T = \{\mathfrak{z} \in \mathbb{C}^n : |z_k| = \varepsilon\}$. Then $P_{\mathfrak{z}_1} = P$ and $\mathfrak{z}_1 \in U$ (if ε is chosen sufficiently small). $\sum_{\nu=0}^{\infty} a_\nu \mathfrak{z}_1^\nu$ converges, therefore $\sum_{\nu=0}^{\infty} |a_\nu \mathfrak{z}_1^\nu|$ also converges. Since $\mathfrak{z}_1 \in \mathring{\mathbb{C}}^n$, $|z_k^{(1)}| \neq 0$ for all k. Therefore every subseries in the above representation at $\mathfrak{z}_1$ also converges absolutely and uniformly in the interior of $P_{\mathfrak{z}_1}$. The limit functions are continuous and are denoted by $\Delta_1, \ldots, \Delta_n$. Since $f(\mathfrak{z}) = f(\mathfrak{z}_0) + z_1 \cdot \Delta_1(\mathfrak{z}) + \cdots + z_n \cdot \Delta_n(\mathfrak{z})$, it follows that f is complex differentiable at $\mathfrak{z}_0$. □

From this proof we obtain the values of the partial derivatives at a point $\mathfrak{z}_0$. For

$$f(\mathfrak{z}) = \sum_{\nu_1, \ldots, \nu_n = 0}^{\infty} a_{\nu_1 \cdots \nu_n}(z_1 - z_1^{(0)})^{\nu_1} \cdots (z_n - z_n^{(0)})^{\nu_n}$$

We obtain

$$f_{z_1}(\mathfrak{z}_0) = a_{1,0,\ldots,0},$$
$$\vdots$$
$$f_{z_n}(\mathfrak{z}_0) = a_{0,\ldots,0,1}.$$

3. The Cauchy Integral

In this section we shall seek additional characterizations of holomorphic functions.

Let $\mathfrak{r} = (r_1, \ldots, r_n)$ be a point in absolute space with $r_\nu \neq 0$ for all ν. Then $P = \{\mathfrak{z} \in \mathbb{C}^n : |z_\nu| < r_\nu \text{ for all } \nu\}$ is a nondegenerate polycylinder about the origin and $T = \{\mathfrak{z} \in \mathbb{C}^n : \tau(\mathfrak{z}) = \mathfrak{r}\}$ is the corresponding distinguished boundary. It will turn out that the values of an arbitrary holomorphic function on P are determined by its values on T.

First of all we must generalize the notion of a complex line integral. Let $K = \{z \in \mathbb{C} : z = re^{i\theta}, r > 0 \text{ fixed}, 0 \leqslant \theta \leqslant 2\pi\}$ be a circle in the complex plane, f a function continuous on K. As usual one writes

$$\int_K f(z)\, dz = \int_0^{2\pi} f(re^{i\theta}) \cdot rie^{i\theta}\, d\theta.$$

The expression on the right is reduced to real integrals by

$$\int_a^b \varphi(t)\, dt : = \int_a^b \operatorname{Re} \varphi(t)\, dt + i \cdot \int_a^b \operatorname{Im} \varphi(t)\, dt.$$

Now let $f = f(\xi)$ be continuous on the n-dimensional torus $T = \{\xi \in \mathbb{C}^n : \tau(\xi) = \mathfrak{r}\}$. Then $h : P \times T \to \mathbb{C}$ with

$$h(\mathfrak{z}, \xi) : = \frac{f(\xi)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)}$$

is also continuous. We define

$$\begin{aligned} F(\mathfrak{z}) : &= \left(\frac{1}{2\pi i}\right)^n \cdot \int_T h(\mathfrak{z}, \xi)\, d\xi_1 \cdots d\xi_n : \\ &= \left(\frac{1}{2\pi i}\right)^n \cdot \int_{|\xi_1| = r_1} \frac{d\xi_1}{\xi_1 - z_1} \int_{|\xi_2| = r_2} \frac{d\xi_2}{\xi_2 - z_2} \cdots \int_{|\xi_n| = r_n} \frac{d\xi_n}{\xi_n - z_n} f(\xi_1, \ldots, \xi_n) \\ &= \left(\frac{1}{2\pi}\right)^n \cdot \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})}{(r_1 e^{i\theta_1} - z_1) \cdots (r_n e^{i\theta_n} - z_n)} \\ &\quad \times r_1 \cdots r_n e^{i(\theta_1 + \cdots + \theta_n)}\, d\theta_1 \cdots d\theta_n. \end{aligned}$$

For each $\mathfrak{z} \in P$, F is well defined and even continuous on P.