

Signal Detection Theory and ROC Analysis

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Signal Detection Theory and ROC Analysis

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To Wilson P. Tanner, Jr.

Preface

This book is intended primarily for the graduate student in experimental psychology who wants to learn about signal detection theory. The student should be aware that signal detection theory has been applied to many diverse fields within experimental psychology. The present treatment does not discuss these applications, nor does it present the results of experiments. My intent is to provide a background with which the student can more profitably read the ever increasing literature on signal detection theory.

The book, *Signal Detection Theory and Psychophysics* by D. M. Green and J. A. Swets (1966, 1974) is a milestone in the history of psychophysics, but many students find some of the important parts rather difficult. I have tried to present the basic material which will help the student of psychology to understand those parts. In addition, I have attempted to cover certain functions and processes, such as the Poisson random process, that are not treated in their book.

The level of mathematical difficulty of this book is intended to be that for a student with one semester of integral calculus and one year of "psychological statistics." However, even after taking a course on integral calculus many students of experimental psychology do not appreciate the importance of such matters as the orthogonal properties of certain trigonometric functions, the transformation of probability density functions, and

other concepts and techniques which are useful, sometimes even necessary, for the understanding of signal detection theory and psychophysics. Although I include little that is new in this treatment, I believe that I have selected certain portions of knowledge in such a fashion as to be particularly useful in the study of signal detection theory. The student has much else to learn and needs all the help available.

This book deals with the decision theory of categorization, including ROC analysis. The properties of ROCs, which are based upon many of the important probability laws used in psychology, are given special emphasis in this part. A succeeding book, *Signal Detection Theory and Ideal Observers*, begins with a discussion of the concept of ideal observers. Then, the minimum essentials of signal analysis are presented. Next, the properties of ensembles of randomly determined waveforms are discussed. (The concept of an ensemble is fundamental in the application of signal detection theory to psychoacoustics, because the best measures of detection, discrimination, and recognition are based upon a series of discrete observation intervals.) Finally, in the second book, a simplified model of Gaussian noise is developed, and this model is used in an analysis of how well an ideal observer can detect, or recognize, signals that are added to this noise.

With time, the treatments of signal detection theory have improved by an emphasis upon the distinction between the relative roles of decision theory and the statistical theory of noise. The present book was written for the purpose of teaching, and therefore I have used various examples, derivations, etc., that were developed by others; I have rewritten this material for the experimental psychologist whenever I felt that such material would be helpful. Acknowledgment of indebtedness to other works is usually given in the text. The bibliography at the end of each chapter lists these works along with others that were helpful to me. These references may help the student as well.

Worked examples were selected in order to illustrate various points, and some of these examples purposely leave out a few steps so that the student may consider them as exercises with each answer provided. There are several appendices, and it is hoped that these additions will be useful to the instructor as well as to the student.

A serious attempt was made to use frequent section numbers in such a way that it would be easy for an instructor to assign certain topics and to omit others. The following sections are marked with an asterisk: 3.6.4, 3.11, 3.11.1, 3.18, 3.18.1, 3.18.2, 4.8, 4.8.1, 4.8.2, 4.8.3, 5.6.4, 5.6.5, 6.2.3, 6.2.6, 7.5.2, 7.5.3, 7.5.4, 7.11, 7.17.3, and 7.17.8. These sections may be omitted without loss of the general theme.

All figures and tables were prepared anew, but, of course, some of them

have appeared in similar form in other treatments of signal detection theory. However, to the best of my knowledge, the chi-square ROCs in Chapter 5, the ROCs based upon geometric distributions in Chapter 6, and certain functions that stem from the Poisson random process in Chapter 7, are novel to the literature on ROC analysis.

Acknowledgments

There are many who have assisted me in the preparation of this book. First of all, as a general matter, the author is happy to acknowledge the help that he has received from his students, the ones with whom he worked at the blackboard by the hour in order to learn of these matters for himself as well as to teach them whatever he could. Then, there are those who read early versions of this section, or that, and they encouraged me to continue the effort. Dr. Frank R. Clarke, Dr. Kent Gummerman, Dr. Dennis McFadden, and Mr. E. G. Pasanen read portions of the manuscript. Also, Mr. William Rankin prevented me from making many errors in the use of the English language. As a student, he showed me many sentences and paragraphs that were too obscure for publication. I thank all these people.

I must express my appreciation to Dr. T. G. Birdsall. During my frequent visits from Bloomington to Ann Arbor from 1956 to 1966, Dr. Birdsall patiently explained to me some of the intricacies of ROC analysis, and he also supported the view that a much idealized model of Gaussian noise is helpful to the beginning student in the understanding of the detection process.

My greatest debt belongs to Dr. David S. Emmerich and to Dr. Neal Viemeister. I have been very fortunate to have these two scholars as my severest critics. Over a period of many years, they have taught me a very great deal. As critics of this book, each of them independently read the complete manuscript with great care, and they pointed out not only my

misuse of commas, but my misuse of certain concepts in mathematics. More than that, both made constructive suggestions which have enriched my treatment of ROC analysis. I am grateful to both of them.

I am pleased that my book was accepted for publication in the series: *Academic Press Series in Cognition and Perception*, edited by Drs. E. C. Carterette and M. Friedman. Professor Carterette's encouragement and constructive suggestions have been very helpful to me.

Professor Wilson P. Tanner, Jr., was the primary force in the introduction of signal detection theory to psychophysics. It is a pleasure to dedicate this little book to Spike Tanner.

Finally, my thanks to my wife Katherine. She has constantly encouraged me, and she has tolerated me in my difficult ways. Above all, she has left me alone when I wanted to work.

There would be many more errors in this book than there are if I had not had the help of all the individuals mentioned above. Of course, I am responsible for all errors that remain, whether they be in my use of arithmetic, grammar, or logic.

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