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Geometry of Algebraic Curves

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Geometry of Algebraic Curves

Volume II
with a contribution by Joseph Daniel Harris

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To the memory of Aldo Andreotti

Preface

This volume is devoted to the foundations of the theory of moduli of algebraic curves defined over the complex numbers. The first volume was almost exclusively concerned with the geometry on a fixed, smooth curve. At the time it was published, the local deformation theory of a smooth curve was well understood, but the study of the geometry of global moduli was in its early stages. This study has since undergone explosive development and continues to do so. There are two reasons for this; one predictable at the time of the first volume, the other not.

The predictable one was the intrinsic algebro-geometric interest in the moduli of curves; this has certainly turned out to be the case. The other is the external influence from physics. Because of this confluence, the subject has developed in ways that are incredibly richer than could have been imagined at the time of writing of Volume I.

When this volume, GAC II, was planned it was envisioned that the centerpiece would be the study of linear series on a general or variable curve, culminating in a proof of the Petri conjecture. This is still an important part of the present volume, but it is not the central aspect. Rather, the main purpose of the book is to provide comprehensive and detailed foundations for the theory of the moduli of algebraic curves. In addition, we feel that a very important, perhaps distinguishing, aspect of GAC II is the blending of the multiple perspectives—algebro-geometric, complex-analytic, topological, and combinatorial—that are used for the study of the moduli of curves.

It is perhaps keeping this aspect in mind that one can understand our somewhat unusual choice of topics and of the order in which they are presented. For instance, some readers might be surprised to see a purely algebraic proof of the projectivity of moduli spaces immediately followed by a detailed introduction to Teichmüller theory. And yet Teichmüller theory is needed for our subsequent discussion of smooth Galois covers of moduli, which in turn is immediately put to use in our approach to the theory of cycles on moduli spaces. Besides, all the above are essential tools in Kontsevich's proof of Witten's conjecture, which is presented in later chapters. Concerning this, the main motivation of our choice of presenting Kontsevich's original proof instead of one of the several more recent ones is—in addition to the great beauty of the proof itself—a desire to be as self-contained as possible. This same desire also motivates in part the presence, at the beginning of the book, of two introductory chapters on the Hilbert scheme and on deformation theory.

In the *Guide for the Reader* we will briefly go through the material we included in this volume. Among the topics we did not cover are the theory of Gromov–Witten invariants, the birational geometry of moduli spaces, the theory of moduli of vector bundles on a fixed curve, the theory of syzygies for the canonical curve, the various incarnations of the Schottky problem together with the related theory of theta function, and the theory of stable rational cohomology of moduli spaces of smooth curves. Some of these topics are covered by excellent publications like [14] for syzygies and [532] for the birational geometry of moduli spaces. On other topics, like the intersection theory of cycles or the theory of the ample cone of moduli spaces of stable curves, we limited ourselves to the foundational material.

Much of Volume I was devoted to the study of the relationship between an algebraic curve and its Jacobian variety. In this volume there is relatively little emphasis on the universal Jacobian or Picard variety and discussion of the moduli of abelian varieties. The latter is a vast and deep subject, especially in its arithmetic aspect, that goes well beyond the scope of this book.

In some instances, important topics, such as the Kodaira dimension of moduli spaces of stable curves, the theory of limit linear series, or the irreducibility of the Severi variety, have appeared elsewhere, specifically in the book *Moduli of Curves* by Joe Harris and Ian Morrison [352]. This is in fact a good opportunity to thank Joe and Ian for their kind words in the introduction of their book. We believe that our respective books complement each other, and we encourage our readers to benefit from their work.

In the bibliographical notes we try to point the reader to the most significant developments, not covered in this volume, of which we were aware at the time of writing. In fact, we view our bibliography and our bibliographical notes as, potentially, an ongoing project.

There is virtually no area in the theory of moduli of curves where the contribution of David Mumford has not been crucial. Our first debt of gratitude is therefore owed to him.

There is a long list of people to whom we would also like to express our gratitude. The first one is Joe Harris, whose generous contribution consists of approximately half of the exercises in this book.

During the long years of preparation of this volume, the following people have greatly contributed with ideas, comments, remarks, and corrections: Gilberto Bini, Alberto Canonaco, Alessandro Chiodo, Herb Clemens, Eduardo Esteves, Domenico Fiorenza, Claudio Fontanari, Jeffrey Giansiracusa, John Harer, Eduard Looijenga, Marco Manetti, Elena Martinengo, Gabriele Mondello, Riccardo Murri, Filippo Natoli, Giuseppe Pareschi, Gian Pietro Pirola, Marzia Polito, Giulia Saccà, Edoardo Sernesi, Roy Smith, Lidia Stoppino, Angelo Vistoli. To all of them we extend our heartfelt sense of gratitude.

We also wish to thank the students in the courses that we taught out of draft versions of parts of the book, who also offered a number of suggestions for improvements.

The first two authors are also grateful to several institutions which hosted them during the preparation of this volume, in particular the Courant Institute of New York University, Columbia University, the Italian Academy in New York, IMPA in Rio de Janeiro, the Institut Henri Poincaré in Paris, the Accademia dei Lincei in Rome, and above all the Institute for Advanced Study. Special thanks go to Enrico Bombieri, who was instrumental in arranging the first two authors' stays at the Institute. It was through his good offices that they were supported on one of these stays as "Sergio Serapioni, Honorary President, Società Trentina Lieviti – Trento (Italy) Members."

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Rome, Pavia, Princeton, 2010

We also want to thank the [illegible] in the courses that we taught and to their [illegible] of parts of the book, who also offered a number of suggestions for improvements.

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[illegible] Paris, Princeton, 2019

Guide for the Reader

The first four chapters of this volume, that is, Chapters IX, X, XI, and XII, are devoted to the construction of the moduli space $\overline{M}_{g,n}$ of stable n-pointed curves of genus g. The three main characters in these chapters are: nodal curves, deformation theory, and Kuranishi families.

Chapter IX gives a self-contained introduction to the Hilbert scheme, explaining the various implications of the concept of flatness and highlighting the case of curves as, for instance, in Mumford's example.

Nodal curves are studied in **Chapter X**. There, we establish the Stable Reduction Theorem (4.11), the theorem on isomorphisms of families of stable curves (5.1), and, in Section 6, the basic constructions of clutching, projection, and stabilization. All these results are fundamental in the construction of the moduli space of stable curves and in the study of its boundary.

The Kodaira–Spencer deformation theory is ubiquitous in this book. Its first appearance is in Section 5 of Chapter IX. It presents itself in its most classical guise as the study of the characteristic system which, in modern terms, translates into the study of the tangent space to the Hilbert scheme. The deformation theory of nodal curves, and in particular of stable ones, is the central theme of **Chapter XI**. There, (5.10) is the key exact sequence describing the tangent space to the local deformation space of a nodal curve. The concept of Kuranishi family is pivotal in the entire volume. The (bases of) Kuranishi families provide the analytic charts for the atlases of moduli stacks of curves. Kuranishi families are constructed by slicing the Hilbert scheme $H_{g,n,\nu}$ of ν-log-canonical embedded stable n-pointed curves of genus g, transversally with respect to the orbits of the natural projective group acting on $H_{g,n,\nu}$, and then restricting to these slices the universal family of curves over $H_{g,n,\nu}$ (see Theorem (6.5) and the key Definitions (6.7) and (6.8) in Chapter XI).

The moduli space $\overline{M}_{g,n}$ is then constructed in **Chapter XII**. We exhibit $\overline{M}_{g,n}$ first as an analytic space, then as an algebraic space, and finally as an orbifold and as a Deligne–Mumford stack. Actually, one of the purposes of this chapter, besides the construction of moduli spaces, and the study of the first properties of their boundary strata, is to give an utilitarian and essentially self-contained introduction to the theory of stacks. This is done in Sections 3–9.

Several topics treated in the first four chapters are not directly aimed at the construction of moduli spaces. Specifically:

- Section 9 of Chapter IX deals with the universal property of the Hilbert scheme with respect to *continuous* families of projective manifolds. Its natural continuation is Section 7 of Chapter XI, where it is shown that the universal property of the Kuranishi family holds also in this context of continuous families of Riemann surfaces. These results will be essential in our presentation of Teichmüller theory in Chapter XV.
- Section 9 of Chapter X is devoted to the Picard–Lefschetz theory of vanishing cycles describing the topological picture of a family of smooth curves degenerating to a nodal one.
- Section 8 of Chapter XI deals with the classical theory of the period map for Riemann surfaces and its infinitesimal behavior.
- In Section 9 of the same chapter we study the positivity properties of the Hodge bundle from the viewpoint of its curvature.
- In the final section of Chapter XI we present Kempf's study of deformations of the symmetric product of a curve leading to the proof of Green's theorem about quadrics passing through the canonical curve (cf. Theorem (4.1) in Chapter VI).

In **Chapter XIII** we present the theory of line bundles on moduli stacks of curves, developing the necessary theory of descent. In the first two sections we introduce the Hodge bundle, the point-bundles $\mathcal{L}_i$, the tangent bundle to the stack $\overline{\mathcal{M}}_{g,n}$, the canonical bundle, the stack divisors corresponding to the codimension one components of its boundary, and the normal bundles to the various boundary strata. The following Section 4 is devoted to the theory of the determinant of the cohomology. This theory is well suited to producing line bundles on moduli stacks, and, at the end of this section, we treat the boundary of moduli as a determinant, leading to important formulae of "restriction to the boundary" as in Lemma (4.22), Proposition (3.10), and formula (4.31). In Section 5 we present the theory of the Deligne pairing, we introduce Mumford's κ_1 class, and we give a concrete version, "without denominators," of the Riemann–Roch theorem for line bundles on families of nodal curves (cf. Theorem (5.31)). In Section 6 we compare the various notions of Picard group for moduli spaces of curves. Section 7 is devoted to Mumford's remarkable idea that the Grothendieck–Riemann–Roch theorem can be effectively used to produce relations among classes in the moduli spaces of curves. There we prove the key formula $\kappa_1 = 12\lambda + \psi - \delta$ for Mumford's class and the formula $K_{\overline{M}_{g,n}} = 13\lambda + \psi - 2\delta - \delta_{1,\emptyset}$ for the canonical class. In the final Section 8 we study the Picard group of the closure $\overline{H}_g \subset \overline{M}_g$ of the hyperelliptic locus.

The fact that $\overline{M}_{g,n}$ is a projective variety (and therefore a scheme) is established in **Chapter XIV**. To prove this we use a mixture of two techniques that are of independent interest. The first one is Mumford's geometric invariant theory. In Sections 2 and 3, we prove the Hilbert–Mumford criterion of stability (Proposition (2.2)), and we use this criterion to prove the stability of the ν-log-canonically embedded *smooth* curves, viewed as points in the appropriate Hilbert scheme. We then take a sharp turn and use stability of smooth curves to find numerical inequalities among cycles in moduli spaces and, consequently, positivity results. Using the same techniques, we then prove the ampleness of Mumford's class κ_1 and hence the projectivity of $\overline{M}_{g,n}$.

Chapter XV gives a self-contained treatment of Teichmüller space and of the modular group. The Teichmüller space $\mathcal{T}_S$ is constructed in Section 2, as a complex manifold, by patching together bases of Kuranishi families. We then examine the natural map $\Phi : B \to \mathcal{T}_S$ from the unit ball, in the space of quadratic differentials on the reference Riemann surface S, to the Teichmüller space $\mathcal{T}_S$. The continuity of this map is an immediate consequence of the results proved in Section 7 of Chapter XI about the universal property of Kuranishi families with respect to continuous families of Riemann surfaces. To prove the injectivity of Φ we first study, in Section 5, the geometry associated to quadratic differentials and then prove, in the following section, Teichmüller's Uniqueness Theorem. As we explain at the end of Section 4, the fact that Φ is a diffeomorphism follows readily from Teichmüller's Uniqueness Theorem and from the elementary theory of the Beltrami equation. In the last section of this chapter we introduce a bordification of Teichmüller space which is very close to the one defined in terms of Fenchel–Nielsen coordinates. Although this bordification is interesting in itself, its only use in our book is in Chapter XIX, where we present Kontsevich's combinatorial expression for the point-bundle classes ψ_i.

Teichmüller space can be thought of as the space representing a rigidification of the moduli functor in which each Riemann surface C comes equipped with a marking (i.e., the homotopy class of a diffeomorphism onto a fixed reference surface). This marking eliminates the automorphism group of C, with the result that Teichmüller space is smooth. The same process of rigidification of the moduli functor can be performed algebraically by considering, for example, pairs consisting of a Riemann surface and the group of points of order n in its Jacobian. More generally, one is looking for finite index normal subgroups Λ of the mapping class group $\Gamma_{g,n}$. Then $\mathcal{T}_{g,n}/\Lambda$ is a Galois cover of $M_{g,n}$ with Galois group $H = \Gamma_{g,n}/\Lambda$. In many instances $\mathcal{T}_{g,n}/\Lambda$ is smooth, so that $M_{g,n}$ can be represented as the quotient of a smooth variety by a finite group H. The main results in this circle of ideas are proved in the first two sections of **Chapter XVI**. When trying to naively push the same ideas to prove analogous results regarding $\overline{M}_{g,n}$, one encounters significant difficulties. These difficulties are addressed in Section 4, and the way to analyze them is to use the Picard–Lefschetz transformation. The problem of expressing $\overline{M}_{g,n}$ as a quotient X/H where X is a smooth variety and H a finite group was solved by Looijenga. In the remaining part of the chapter we present a variation of Looijenga construction due to Abramovich, Corti, and

Vistoli which exhibits X as a fine moduli space, in fact as a moduli space for admissible G-covers, where G is an appropriate finite group, and H is a quotient of the semidirect product $G^n \ltimes \mathrm{Aut}(G)$.

The fact that $\overline{M}_{g,n}$ can be expressed as the quotient X/H, with X a smooth variety and H a finite group, makes it relatively easy to talk about its Chow rings. The theory of cycles in $\overline{M}_{g,n}$ is the central subject of **Chapter XVII**. After presenting, in Section 2, the foundational material on the intersection theory of stacks of the form $[X/H]$ with X smooth and H a finite group, in Section 3 we introduce the tautological classes. These are the Mumford–Morita–Miller classes (i.e., the κ-classes), the point-bundle classes (i.e., the ψ-classes), the Hodge classes (i.e, the λ-classes), and the boundary classes (i.e., the δ-classes). In Section 4 we describe the behavior of these classes under push-forward and pullback via the projection morphism $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ and the clutching morphisms $\xi_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}$ from the various boundary strata. In Section 5, following Mumford, we use, on the one hand, Grothendieck's Riemann–Roch theorem to find relations between the Hodge classes and the κ classes, and on the other hand, using the flatness of the Gauss–Manin connection, we exhibit a set of generators for the tautological ring $R^\bullet(M_g)$ (i.e., the ring generated by the tautological classes). At the end of the section we discuss Deligne's canonical extension of the Gauss–Manin connection to the boundary of moduli. Section 6 offers a brief and informal discussion of the tautological ring, presenting two results, a nonvanishing theorem for the tautological class κ_{g-2} due to Faber, and a vanishing theorem for polynomials of degree greater than $g-2$ in the tautological classes, due to Looijenga. Both results are proved in subsequent parts of the book. In the last section we present Keel's result on the Chow ring of $\overline{M}_{0,n}$, and we give a direct computation of $A^1(\overline{M}_{0,n})$.

The fact that $M_{g,n}$ is a rational $K(\Gamma_{g,n}, 1)$ for the mapping class group $\Gamma_{g,n}$ hints to the possibility of studying $M_{g,n}$ from a combinatorial point of view. This is done in **Chapter XVIII**, where we introduce a $\Gamma_{g,n}$-invariant triangulation of the Teichmüller space $\mathcal{T}_{g,n}$. Loosely speaking, the complex structure on a Riemann surface determines (and is determined by) a graph embedded in the Riemann surface itself. This makes it possible to give a $\Gamma_{g,n}$-invariant cellular decomposition of $\mathcal{T}_{g,n}$ where the cells are labelled by these graphs. In the first two sections we introduce the arc system complex and, by duality, the ribbon graphs, which are the basic tools for the combinatorial description of $\mathcal{T}_{g,n}$. To prove that (a subcomplex of) the arc system complex gives a combinatorial model of $\mathcal{T}_{g,n}$, one may choose either the theory of Jenkins–Strebel differentials, or alternatively, via uniformization, the canonical hyperbolic metric on Riemann surfaces. We choose the latter since it more easily enables one to extend the cellular decomposition to the bordification of $\mathcal{T}_{g,n}$. After explaining, in Section 4, how hyperbolic geometry is used to obtain the cellular decomposition of $\mathcal{T}_{g,n}$ and after recalling, in Sections 5 and 6, some basic facts about the uniformization theorem and the Poincaré metric, in Sections 7 and 8 we give the construction of the cellular decomposition of $\mathcal{T}_{g,n}$. In this book the cellular decomposition of moduli spaces is used in two ways. First of all to give a simple and direct proof of the vanishing of the rational homology of $M_{g,n}$ in high degree. These applications are given in Chapter XIX. The second enters when computing the intersection number of tautological classes in Kontsevich's proof of Witten conjecture, which is given in Chapter XX. In fact, this last application requires that the cellular decomposition of $M_{g,n}$ be extended to a suitable compactification of moduli space. This task, which is technically more demanding, is carried out in Sections 9–12.

Chapter XIX discusses the first consequences of the cellular decomposition constructed in Chapter XVIII. We begin by computing the rational cohomology of $\overline{M}_{g,n}$ in degrees one and two. This computation can be performed by elementary methods by virtue of the vanishing of the high homology of $M_{g,n}$ which, in turn, is a direct consequence of the cellular decomposition. This is carried out in Sections 2, 3, and 4. In Section 5, after a very brief discussion of Harer's stability theorem and of the Madsen–Weiss and Tillmann theorems on the stable rational cohomology of $M_{g,n}$, we prove Harer's theorem on the second homology of $M_{g,n}$. This we do by using the knowledge of $H^2(\overline{M}_{g,n};\mathbb{Q})$ and Deligne's spectral sequence for the complement of a divisor with normal crossings. Further uses of the cellular decomposition are presented in Section 7, where we give Kontsevich's combinatorial expression for the point-bundle classes ψ, and in Section 8, where we give Kontsevich's combinatorial expression for an orientation form on $M_{g,n}$.

Chapter XX is almost entirely devoted to Kontsevich's proof of Witten's conjecture on the intersection numbers of the ψ-classes. The proof is self-contained, with the exception of an algebraic result by Itzykson for which there is a very clear and well-written reference. In the first two sections we review Witten's generating series for the intersection numbers of the ψ-classes, introduce the Virasoro operators, and describe their link with the KdV hierarchy. In Section 4 we prove Kontsevich's combinatorial formula expressing Witten's generating series as a sum over ribbon graphs. We then give a self-contained treatment of the Feynman diagram expansion of matrix integrals, and finally, in Section 6, we express Kontsevich's combinatorial sum as a matrix integral and, using this, conclude the proof of Witten's conjecture. As we show in Section 7, the knowledge of the intersection numbers of the ψ-classes can be used to prove the nonvanishing of the class κ_{g-2}. This result by Faber gives the threshold for the non-vanishing of the tautological ring of M_g. In fact, in Section 4 of Chapter XXI we prove a theorem by Looijenga stating that the ring of tautological classes on M_g vanishes in degree strictly larger than $g-2$. After recalling some basic facts about equivariant cohomology, in the last two sections of Chapter XX we present Harer and Zagier's computation of the virtual Euler–Poincaré characteristic of $M_{g,n}$.

The Brill–Noether theory is one the central themes of the first volume of this book. There we study the static aspect of this theory, namely the theory of special linear series on a fixed curve. In our final **Chapter XXI** we study the Brill–Noether theory for smooth curves moving with moduli. In the first few sections, aside for an intermission in which we prove the vanishing theorem of Looijenga we mentioned above, we construct the basic varieties of the Brill–Noether theory for smooth moving curves, and we describe their tangent spaces in terms of the fundamental homomorphisms $\mu_0 : H^0(C;L)\otimes H^0(C;\omega_C L^{-1}) \to H^0(\omega_C)$ and $\mu_1 : \ker\mu_0 \to H^0(\omega_C^2)$, where C is a smooth curve and L a line bundle on it. We also connect these maps to the normal sheaf relative to the morphism $\phi_L : C \to \mathbb{P}^r$, where $r = h^0(C,L)-1$. In Section 7 we present Lazarsfeld's elegant proof of Petri's conjecture. In the remaining part of the chapter we concentrate mostly on the study of g^1_d's and g^2_d's on smooth curves. We revisit a number of classical results and present some nonclassical ones, related to, among others, the Hurwitz scheme, the Severi variety of plane curves of given degree and genus and the unirationality of M_g for small values of g.

Notational conventions and blanket assumptions

- Unless otherwise stated, all schemes are implicitly assumed to be of finite type over $\mathbb{C}$.
- If V is a vector space or a vector bundle, $\mathbb{P}V$ is the projective space, or projective bundle, of *lines* in V, or in the fibers of V.
- If $\varphi : X \to S$ is a morphism of schemes or of analytic spaces and T is a locally closed subscheme or subspace of S, we write X_T to denote the fiber product $X \times_S T$. Likewise, if s is a point of S, we write X_s to denote the fiber $\varphi^{-1}(s)$.
- We usually write $\mathrm{Sym}^q V$ to indicate the q-th symmetric product of the module or coherent sheaf V. Occasionally, we instead use the notation S^qV, especially when V is a vector space.