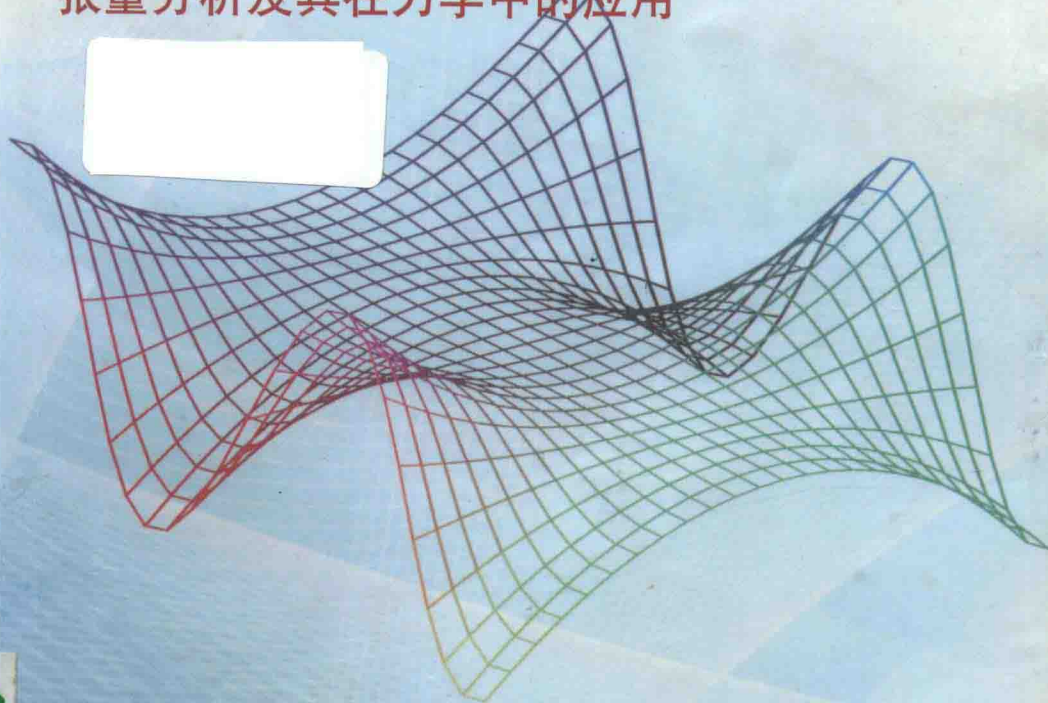


# Tensor Analysis with Applications in Mechanics

张量分析及其在力学中的应用



Leonid P Lebedev  
Michael J Cloud  
Victor A Eremeyev

World Scientific

世界图书出版公司  
[www.wpcbj.com.cn](http://www.wpcbj.com.cn)

# Tensor Analysis with Applications in Mechanics

**Leonid P. Lebedev**

*National University of Colombia*

*Southern Federal University, Russia*

**Michael J. Cloud**

*Lawrence Technological University, USA*

**Victor A. Eremeyev**

*Martin-Luther University Halle-Wittenberg, Germany*

*Southern Scientific Center of Russian Academy of Science*

*Southern Federal University, Russia*

 **World Scientific**

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

*Published by*

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

**British Library Cataloguing-in-Publication Data**

A catalogue record for this book is available from the British Library.

**TENSOR ANALYSIS WITH APPLICATIONS IN MECHANICS**

Copyright © 2010 by World Scientific Publishing Co. Pte. Ltd.

*All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.*

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

ISBN-13 978-981-4313-12-4

ISBN-10 981-4313-12-2

Reprint arranged with World Scientific Co. Pte. Ltd., Singapore.

本书由新加坡 World Scientific Publishing Co. (世界科技出版公司) 授权重印出版, 限于中国大陆地区发行。

## 图书在版编目 (CIP) 数据

张量分析及其在力学中的应用 = Tensor Analysis with Applications in Mechanics :  
英文/(俄)乐博德夫(Lebedev, L. P.)著. —影印本. —北京:世界图书出版公司  
北京公司, 2014. 8

ISBN 978 - 7 - 5100 - 8453 - 9

I. ①张… II. ①乐… III. ①张量分析—应用—力学—研究—英文 IV. ① 03

中国版本图书馆 CIP 数据核字 (2014) 第 185714 号

---

书 名: Tensor Analysis with Applications in Mechanics  
作 者: Leonid P Lebedev, Michael J. Cloud, Victor A Eremeyev  
中译名: 张量分析及其在力学中的应用  
责任编辑: 高蓉 刘慧

---

出 版 者: 世界图书出版公司北京公司  
印 刷 者: 三河市国英印务有限公司  
发 行 者: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)  
联系电话: 010 - 64021602, 010 - 64015659  
电子信箱: kjb@wpcbj.com.cn

---

开 本: 24 开  
印 张: 16.5  
版 次: 2015 年 1 月  
版权登记: 图字: 01 - 2013 - 5770

---

书 号: 978 - 7 - 5100 - 8453 - 9 定 价: 69.00 元

---



# **Tensor Analysis with Applications in Mechanics**



# Foreword

Every science elaborates tools for the description of its objects of study. In classical mechanics we make extensive use of vectorial quantities: forces, moments, positions, velocities, momenta. Confining ourselves to a single coordinate frame, we can regard a vector as a fixed column matrix. The definitive trait of a vector quantity, however, is its objectivity; a vector does not depend on our choice of coordinate frame. This means that as soon as the components of a force are specified in one frame, the components of that force relative to any other frame can be found through the use of appropriate transformation rules.

But vector quantities alone do not suffice for the description of continuum media. The stress and strain at a point inside a body are also objective quantities; however, the specification of each of these relative to a given frame requires a square matrix of elements. Under changes of frame, these elements transform according to rules different from the transformation rules for vectors. Stress and strain tensors are examples of tensors of the second order. We could go on to cite other objective quantities that occur in the mechanics of continua. The set of elastic moduli associated with Hooke's law comprise a tensor of the fourth order; as such, these moduli obey yet another set of transformation rules. Despite the differences that exist between the transformation laws for the various types of objective quantities, they all fit into a unified scheme: the theory of tensors.

Tensor theory not only relieves our memory from a huge burden, but enables us to carry out differential operations with ease. This is the case even in curvilinear coordinate systems. Through the unmatched simplicity and brevity it affords, tensor analysis has attained the status of a general language that can be spoken across the various areas of continuum physics. A full comprehension of this language has become necessary for those working



in electromagnetism, the theory of relativity, or virtually any other field-theoretic discipline. More modern books on physical subjects invariably contain appendices in which various vector and tensor identities are listed. These may suffice when one wishes to verify the steps in a development, but can leave one in doubt as to how the facts were established or, *a fortiori*, how they could be adapted to other circumstances. On the other hand, a comprehensive treatment of tensors (e.g., involving a full excursion into multilinear algebra) is necessarily so large as to be flatly inappropriate for the student or practicing engineer.

Hence the need for a treatment of tensor theory that does justice to the subject and is friendly to the practitioner. The authors of the present book have met these objectives with a presentation that is simple, clear, and sufficiently detailed. The concise text explains practically all those formulas needed to describe objects in three-dimensional space. Occurrences in physics are mentioned when helpful, but the discussion is kept largely independent of application area in order to appeal to the widest possible audience. A chapter on the properties of curves and surfaces has been included; a brief introduction to the study of these properties can be considered as an informative and natural extension of tensor theory.

I.I. Vorovich

Late Professor of Mechanics and Mathematics

Rostov State University, Russia

Fellow of Russian Academy of Sciences

(1920–2001)

# Preface

The first edition of this book was written for students, engineers, and physicists who must employ tensor techniques. We did not present the material in complete generality for the case of  $n$ -dimensional space, but rather presented a three-dimensional version (which is easy to extend to  $n$  dimensions); hence we could assume a background consisting only of standard calculus and linear algebra.

We have decided to extend the book in a natural direction, adding two chapters on applications for which tensor analysis is the principal tool. One chapter is on linear elasticity and the other is on the theory of shells and plates. We present complete derivations of the equations in these theories, formulate boundary value problems, and discuss the problem of uniqueness of solutions, Lagrange's variational principle, and some problems on vibration. Space restrictions prohibited us from presenting an entire course on mechanics; we had to select those questions in elasticity where the role of tensor analysis is most crucial.

We should mention the essential nature of tensors in elasticity and shell theory. Of course, to solve a certain engineering problem, one should write things out in component form; sometimes this takes a few pages. The corresponding formulas in tensor notation are quite simple, allowing us to grasp the underlying ideas and perform manipulations with relative ease. Because tensor representation leads quickly and painlessly to component-wise representation, this technique is ideal for presenting continuum theories to students.

The first five chapters are largely unmodified, aside from some new problem sets and material on tensorial functions needed for the chapters on elasticity. The end-of-chapter problems are supplementary, whereas the integrated exercises are required for a proper understanding of the text.

In the first edition we used the term *rank* instead of *order*. This was common in the older literature. In the newer literature, the term “rank” is often assigned a different meaning.

Because the book is largely self-contained, we make no attempt at a comprehensive reference list. We merely list certain books that cover similar material, that extend the treatment slightly, or that may be otherwise useful to the reader.

We are deeply grateful to our World Scientific editor, Mr. Tjan Kwang Wei, for his encouragement and support.

L.P. Lebedev  
*Department of Mathematics*  
*National University of Colombia, Colombia*

M.J. Cloud  
*Department of Electrical and Computer Engineering*  
*Lawrence Technological University, USA*

V.A. Eremeyev  
*South Scientific Center of RASci*  
§  
*Department of Mathematics, Mechanics*  
*and Computer Sciences*  
*South Federal University, Russia*

## **Preface to the First Edition**

Originally a vector was regarded as an arrow of a certain length that could represent a force acting on a material point. Over a period of many years, this naive viewpoint evolved into the modern interpretation of the notion of vector and its extension to tensors. It was found that the use of vectors and tensors led to a proper description of certain properties and behaviors of real natural objects: those aspects that do not depend on the coordinate systems we introduce in space. This independence means that if we define such properties using one coordinate system, then in another system we can recalculate these characteristics using valid transformation rules. The ease

with which a given problem can be solved often depends on the coordinate system employed. So in applications we must apply various coordinate systems, derive corresponding equations, and understand how to recalculate results in other systems. This book provides the tools necessary for such calculation.

Many physical laws are cumbersome when written in coordinate form but become compact and attractive looking when written in tensorial form. Such compact forms are easy to remember, and can be used to state complex physical boundary value problems. It is conceivable that soon an ability to merely formulate statements of boundary value problems will be regarded as a fundamental skill for the practitioner. Indeed, computer software is slowly advancing toward the point where the only necessary input data will be a coordinate-free statement of a boundary value problem; presumably the user will be able to initiate a solution process in a certain frame and by a certain method (analytical, numerical, or mixed), or simply ask the computer algorithm to choose the best frame and method. In this way, vectors and tensors will become important elements of the macro-language for the next generation of software in engineering and applied mathematics.

We would like to thank the editorial staff at World Scientific — especially Mr. Tjan Kwang Wei and Ms. Sook-Cheng Lim — for their assistance in the production of this book. Professor Byron C. Drachman of Michigan State University commented on the manuscript in its initial stages. Lastly, Natasha Lebedeva and Beth Lannon-Cloud deserve thanks for their patience and support.

L.P. Lebedev

*Department of Mechanics and Mathematics  
Rostov State University, Russia*

✉

*Department of Mathematics  
National University of Colombia, Colombia*

M.J. Cloud

*Department of Electrical and Computer Engineering  
Lawrence Technological University, USA*



# Contents

<i>Foreword</i>	v
<i>Preface</i>	vii
<b>Tensor Analysis</b>	<b>1</b>
1. Preliminaries	3
1.1 The Vector Concept Revisited . . . . .	3
1.2 A First Look at Tensors . . . . .	4
1.3 Assumed Background . . . . .	5
1.4 More on the Notion of a Vector . . . . .	7
1.5 Problems . . . . .	9
2. Transformations and Vectors	11
2.1 Change of Basis . . . . .	11
2.2 Dual Bases . . . . .	12
2.3 Transformation to the Reciprocal Frame . . . . .	17
2.4 Transformation Between General Frames . . . . .	18
2.5 Covariant and Contravariant Components . . . . .	21
2.6 The Cross Product in Index Notation . . . . .	22
2.7 Norms on the Space of Vectors . . . . .	24
2.8 Closing Remarks . . . . .	27
2.9 Problems . . . . .	27
3. Tensors	29
3.1 Dyadic Quantities and Tensors . . . . .	29

3.2	Tensors From an Operator Viewpoint . . . . .	30
3.3	Dyadic Components Under Transformation . . . . .	34
3.4	More Dyadic Operations . . . . .	36
3.5	Properties of Second-Order Tensors . . . . .	40
3.6	Eigenvalues and Eigenvectors of a Second-Order Symmetric Tensor . . . . .	44
3.7	The Cayley–Hamilton Theorem . . . . .	48
3.8	Other Properties of Second-Order Tensors . . . . .	49
3.9	Extending the Dyad Idea . . . . .	56
3.10	Tensors of the Fourth and Higher Orders . . . . .	58
3.11	Functions of Tensorial Arguments . . . . .	60
3.12	Norms for Tensors, and Some Spaces . . . . .	66
3.13	Differentiation of Tensorial Functions . . . . .	70
3.14	Problems . . . . .	77
4.	Tensor Fields . . . . .	85
4.1	Vector Fields . . . . .	85
4.2	Differentials and the Nabla Operator . . . . .	94
4.3	Differentiation of a Vector Function . . . . .	98
4.4	Derivatives of the Frame Vectors . . . . .	99
4.5	Christoffel Coefficients and their Properties . . . . .	100
4.6	Covariant Differentiation . . . . .	105
4.7	Covariant Derivative of a Second-Order Tensor . . . . .	106
4.8	Differential Operations . . . . .	108
4.9	Orthogonal Coordinate Systems . . . . .	113
4.10	Some Formulas of Integration . . . . .	117
4.11	Problems . . . . .	119
5.	Elements of Differential Geometry . . . . .	125
5.1	Elementary Facts from the Theory of Curves . . . . .	126
5.2	The Torsion of a Curve . . . . .	132
5.3	Frenet–Serret Equations . . . . .	135
5.4	Elements of the Theory of Surfaces . . . . .	137
5.5	The Second Fundamental Form of a Surface . . . . .	148
5.6	Derivation Formulas . . . . .	153
5.7	Implicit Representation of a Curve; Contact of Curves . . . . .	156
5.8	Osculating Paraboloid . . . . .	162
5.9	The Principal Curvatures of a Surface . . . . .	164

5.10	Surfaces of Revolution . . . . .	168
5.11	Natural Equations of a Curve . . . . .	170
5.12	A Word About Rigor . . . . .	173
5.13	Conclusion . . . . .	175
5.14	Problems . . . . .	175

## Applications in Mechanics 179

6.	Linear Elasticity . . . . .	181
6.1	Stress Tensor . . . . .	181
6.2	Strain Tensor . . . . .	190
6.3	Equation of Motion . . . . .	193
6.4	Hooke's Law . . . . .	194
6.5	Equilibrium Equations in Displacements . . . . .	200
6.6	Boundary Conditions and Boundary Value Problems . . . . .	202
6.7	Equilibrium Equations in Stresses . . . . .	203
6.8	Uniqueness of Solution for the Boundary Value Problems of Elasticity . . . . .	205
6.9	Betti's Reciprocity Theorem . . . . .	206
6.10	Minimum Total Energy Principle . . . . .	208
6.11	Ritz's Method . . . . .	216
6.12	Rayleigh's Variational Principle . . . . .	221
6.13	Plane Waves . . . . .	227
6.14	Plane Problems of Elasticity . . . . .	230
6.15	Problems . . . . .	232
7.	Linear Elastic Shells . . . . .	237
7.1	Some Useful Formulas of Surface Theory . . . . .	239
7.2	Kinematics in a Neighborhood of $\Sigma$ . . . . .	242
7.3	Shell Equilibrium Equations . . . . .	244
7.4	Shell Deformation and Strains; Kirchhoff's Hypotheses . . . . .	249
7.5	Shell Energy . . . . .	256
7.6	Boundary Conditions . . . . .	259
7.7	A Few Remarks on the Kirchhoff-Love Theory . . . . .	261
7.8	Plate Theory . . . . .	263
7.9	On Non-Classical Theories of Plates and Shells . . . . .	277

Appendix A	Formulary . . . . .	287
------------	---------------------	-----



Appendix B Hints and Answers	315
<i>Bibliography</i>	355
<i>Index</i>	359