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HANDBOOK OF GLOBAL ANALYSIS

Edited by

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Handbook of Global Analysis

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Preface

The group of topics known broadly as 'Global Analysis' has developed considerably over the past twenty years, to such an extent that workers in one area may sometimes be unaware of relevant results from an adjacent area. The many variations in notation and terminology add to the difficulty of comparing one branch of the subject with another.

Our purpose in preparing this Handbook has been to try to overcome these difficulties by presenting a collection of articles which, together, give an overall survey of the subject. We have been guided in this task by the MSC2000 classification, and so the scope of the Handbook may be described by saying that it covers the 58-XX part of the classification: ranging from the structure of manifolds, through the vast area of partial differential equations, to particular topics with their own distinctive flavour such as holomorphic bundles, harmonic maps, variational calculus and non-commutative geometry. The coverage is not complete, but we hope that it is sufficiently broad to provide a useful reference for researchers throughout global analysis, and that it will also be of benefit to mathematical physicists and to PhD and post-doctoral students in both areas.

The main work involved in the preparation of the Handbook has, of course, been that of the authors of the articles, who have carried out their task with skill and professionalism. Our debt to them is immediate and obvious. Some other potential authors have, for personal reasons, been unable to offer contributions to the Handbook, but we hope that those omissions will not detract too much from its value. The editors also wish to acknowledge the assistance of Petr Volný in the formatting of the LaTeX manuscripts, and of Andy Deelen, Kristi Green and Simon Pepping at Elsevier for their help and advice during the preparation of the book. In addition, we should like to record our particular thanks to Arjen Sevenster from Elsevier, who commissioned the project and gave us support and encouragement during its development.

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Demeter Krupka and David Saunders

Global aspects of Finsler geometry[1]

Tadashi Aikou and László Kozma

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1 Finsler metrics and connections

1.1 Finsler metrics

Let $\pi : TM \to M$ be the tangent bundle of a connected smooth manifold M of $\dim M = n$. We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_xM$. We denote by $z(M)$ the zero section of TM, and by $TM^\times$ the slit tangent bundle $TM \backslash z(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, \cdots, x^n)$. By setting $v = \sum y^i \left(\partial/\partial x^i\right)_x$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^n)$ on $\pi^{-1}(U)$.

Definition 1.1 A function $F : TM \longrightarrow \mathbb{R}$ is called a *Finsler metric* on M if

(1) $F(x, y) \geq 0$, and $F(x, y) = 0$ if and only if $y = 0$,

(2) $F(x, \lambda y) = \lambda F(x, y)$ for $\forall \lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$,

(3) $F(x, y)$ is smooth on $TM^\times$, the out-side of the zero section,

[1] Tadashi Aikou: work supported in part by Grant-in-Aid for Scientific Research No. 17540086(2006), The Ministry of Education, Science Sports and Culture. László Kozma: partially supported by the Hungarian Scientific Research Fund OTKA T048878.

(4) $G = F^2/2$ is *strictly convex* on each tangent space T_xM, that is, the Hessian (G_{ij}) defined by

$$G_{ij}(x, y) = \frac{\partial^2 G}{\partial y^i \partial y^j} \tag{1.1}$$

is positive-definite,

are satisfied. The pair (M, F) is called a *Finsler manifold.*

We note that the last condition in this definition is equivalent to the convexity of the unit ball $B_x = \{y \in T_xM \mid F(x, y) \leq 1\}$.

If a Finsler metric F is defined, then the norm $\|y\|$ of each $y \in T_xM$ is defined by $\|y\| = F(x, y)$, and the length $s(t)$ of a smooth curve $c(t) = (x^1(t), \cdots, x^n(t))$ is defined by $s(t) = \int_0^1 \|\dot{c}(t)\| dt = \int_0^1 F(x(t), \dot{x}(t))\, dt$.

Example 1.2 *(Funk metric)* Let g be a Riemannian metric on M. We define $\alpha : TM \to \mathbb{R}$ by $\alpha(v) = \sqrt{g(v, v)}$. Since α is convex, there exists a 1-from β such that $\beta(v) \leq \alpha(v)$. The function $F = \alpha + \beta$ defines a convex Finsler metric on M so-called *Randers metric.* We shall review a typical example of Randers metric (see [37] or [14]). Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the standard coordinate $(x^1, \cdots, x^n)$, and $\mathbb{B}$ the unit ball centered the origin: $\mathbb{B} = \{x \in \mathbb{R}^n \mid \phi(x) = 1 - \|x\|^2 > 0\}$. The Riemannian metric g_H defined by

$$g_H = \frac{(1 - \|x\|^2)\left(\sum dx^i\right)^2 + \left(\sum x^i dx^i\right)^2}{(1 - \|x\|^2)^2}$$

is called the *Hilbert metric* on $\mathbb{B}$. We define a 1-form β by

$$\beta = \frac{\sum x^i dx^i}{1 - \|x\|^2} = -\frac{1}{2} d \log \phi.$$

The norm $\|\beta\|_H$ of β with respect to g_H is given by $\|\beta(x)\|_H = \|x\| < 1$, and thus the function F on $\mathbb{B}$ defined by $F(v) = \sqrt{g_H(v, v)} + \beta(v)$ is a Finsler metric called the *Funk metric* on $\mathbb{B}$. We note that the relation between g_H and F is given by

$$\|v\|_H = \frac{1}{2}\left[F(v) + F(-v)\right]$$

for all $v \in TM$.

For the differential π_* of the submersion $\pi : TM^\times \longrightarrow M$, the *vertical subbundle* V of $T(TM^\times)$ is defined by $V = \ker \pi_*$, and V is locally spanned by $\{\partial/\partial y^1, \cdots, \partial/\partial y^n\}$ on each $\pi^{-1}(U)$. Then it induces the exact sequence

$$0 \longrightarrow V \overset{i}{\longrightarrow} T(TM^\times) \overset{\pi_*}{\longrightarrow} \widetilde{TM} \longrightarrow 0, \tag{1.2}$$

where $\widetilde{TM} = \{(y, v) \in TM^\times \times TM \,\big|\, v \in T_{\pi(y)}M\}$ is the pull-back bundle π^*TM.

$$\begin{array}{ccc} \widetilde{TM} & \longrightarrow & TM \\ \downarrow & & \downarrow \pi \\ TM^\times & \underset{\pi}{\longrightarrow} & M \end{array}$$

Since the natural local frame field $\{\partial/\partial x^i\}_{i=1,\cdots,n}$ on each U is identified with the one of $\widetilde{TM}$ on $\pi^{-1}(U)$, any section X of $\widetilde{TM}$ is written in the form $X = \sum(\partial/\partial x^i) \otimes X^i$ for smooth functions X^i on each $\pi^{-1}(U)$. Furthermore, since $\ker \pi_* = V$, the differential π_* is given by $\pi_* = \sum(\partial/\partial x^i) \otimes dx^i$.

We define a metric G on the bundle $\widetilde{TM}$ by

$$G(X, Y) = \sum G_{ij} X^i Y^j \tag{1.3}$$

for every section $X = \sum(\partial/\partial x^i) \otimes X^i$ and $Y = \sum(\partial/\partial x^j) \otimes Y^j$. We also set

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i\,\partial y^j\,\partial y^k}.$$

Then we define a symmetric tensor field $C : \otimes^3 \widetilde{TM} \to \mathbb{R}$ by

$$C(X, Y, Z) = \sum C_{ijk} X^i Y^j Z^k \tag{1.4}$$

for all sections X, Y, Z of $\widetilde{TM}$. It is trivial that C vanishes identically if and only if G is a Riemannian metric on M. This tensor field C is called the *Cartan tensor field.*

In the sequel, we use the notation $A^0(\widetilde{TM})$ for the space of smooth sections of $\widetilde{TM}$. Since $\widetilde{TM}$ is naturally identified with $V \cong \ker \pi_*$, any section X of $\widetilde{TM}$ is considered as a section of V. We denote by X^V the section of V corresponding to $X \in A^0(\widetilde{TM})$:

$$A^0(\widetilde{TM}) \ni X = \sum \frac{\partial}{\partial x^i} \otimes X^i \quad \Longleftrightarrow \quad \sum \frac{\partial}{\partial y^i} \otimes X^i := X^V \in A^0(V).$$

The following is trivial since (1.2) is exact.

$$\pi_*(X^V) = 0 \tag{1.5}$$

for every $X \in A^0(\widetilde{TM})$.

The multiplier group $\mathbb{R}^+ \cong \{cI \in GL(n, \mathbb{R});\ c \in \mathbb{R}^+\} \subset GL(n, \mathbb{R})$ acts on the total space by multiplication

$$m_\lambda : TM^\times \ni v = (x, y) \to \lambda v = (x, \lambda y) \in TM^\times$$

for every $\lambda \in \mathbb{R}^+$. This action induces a canonical section $\mathcal{E}$ of V defined by $\mathcal{E}(v) = (v, v)$ for all $v \in TM^\times$. By the homogeneity of F, we have

$$\mathcal{E}(F) = \left.\frac{d}{dt}\right|_{t=0} F(x, y + t\mathcal{E}) = F.$$

We shall consider $\mathcal{E}$ as a section of $\widetilde{TM}$, and we denote it by the same notation $\mathcal{E}$, that is, $\mathcal{E}(x, y) = \sum(\partial/\partial x^i) \otimes y^i$. This section $\mathcal{E}$ is called the *tautological section* of $\widetilde{TM}$. Then it is easily shown that $F = \sqrt{G(\mathcal{E}, \mathcal{E})}$ and

$$C(\mathcal{E}, \bullet, \bullet) \equiv 0. \tag{1.6}$$

1.2 Ehresmann connection

For the submersion $\pi : TM^\times \to M$, the vertical subbundle is defined by $V = \ker \pi_*$, while the *horizontal subbundle* H is defined by a subbundle $H \subset T(TM^\times)$ which is complementary to V. These subbundles give a smooth splitting

$$T(TM^\times) = H \oplus V. \tag{1.7}$$

Although the vertical subbundle V is uniquely determined, the horizontal subbundle is not canonically determined. An *Ehresmann connection* of the submersion $\pi : TM^\times \to M$ is a selection of horizontal subbundles. In this report, we shall define this as follows.

Definition 1.3 An *Ehresmann connection* of the submersion $\pi : TM^\times \to M$ is a bundle morphism $\theta : T(TM^\times) \to \widetilde{TM}$ satisfying

$$\theta(X^V) = X \tag{1.8}$$

for every $X \in A^0(\widetilde{TM})$.

If an Ehresmann connection θ is given, then a horizontal subbundle H is defined by $H = \ker \theta$. In this report, we shall assume that the subbundle H defined by θ is invariant by the action $m_\bullet$, that is, $(m_\lambda)_* H = H \circ m_\lambda$ for all $\lambda \in \mathbb{R}^+$. This assumption is equivalent to

$$\mathcal{L}_\mathcal{E} H \subset H. \tag{1.9}$$

Remark 1.4 A *linear connection* of the tangent bundle TM is a selection of horizontal subbundles in $GL(n, \mathbb{R})$-invariant way. Thus, an Ehresmann connection θ in our sense is sometimes called a *non-linear connection* of TM.

In the sequel, we denote by A^k and $A^k(\widetilde{TM})$ the space of smooth k-forms and $\widetilde{TM}$-valued k-form on $TM^\times$ respectively. We suppose that an Ehresmann connection θ is given. Then, the exterior differential $d : A^k \to A^{k+1}$ is decomposed into the form $d = d^H \oplus d^V$ according to the decomposition (1.7), where d^H is the differential along H and d^V is the one along V. If a covariant derivation $D : A^0(\widetilde{TM}) \to A^1(\widetilde{TM})$ of the bundle $\widetilde{TM}$ is also decomposed into the form $D = D^H \oplus D^V$.

Proposition 1.5 *If an Ehresmann connection θ is given, then there exists a covariant exterior derivation D of $\widetilde{TM}$ satisfying*

$$\theta = D\mathcal{E}, \tag{1.10}$$

or equivalently

$$D^H \mathcal{E} = 0. \tag{1.11}$$

Proof We define a covariant derivation D by $D^V = d^V$ and $D^H_X Y = \theta[X^H, Y^V]$ for all $X, Y \in A^0(\widetilde{TM})$. It is easily shown that $D = D^H \oplus d^V$ is a covariant derivation on $\widetilde{TM}$. Then we have

$$D^V_X \mathcal{E} = X^V(\mathcal{E}) = X^V = \theta(X)$$

and, from (1.9) we obtain

$$D^H_X \mathcal{E} = \theta[X^H, \mathcal{E}] = -\theta(\mathcal{L}_\mathcal{E} X^H) = 0.$$

Therefore we obtain (1.11). □

Since we are concerned with the tangent bundle, $\widetilde{TM}$ is also naturally identified with the horizontal subbundle H, and any section X of $\widetilde{TM}$ is considered as a section of H. We denote by X^H the section of H corresponding to $X \in A^0(\widetilde{TM})$:

$$A^0(\widetilde{TM}) \ni X = \sum \frac{\partial}{\partial x^i} \otimes X^i \quad \Longleftrightarrow \quad \sum \frac{\delta}{\delta x^i} \otimes X^i := X^H \in A^0(H),$$

where

$$\left\{ \frac{\delta}{\delta x^1} = \left(\frac{\partial}{\partial x^1} \right)^H, \cdots, \frac{\delta}{\delta x^n} = \left(\frac{\partial}{\partial x^n} \right)^H \right\}$$

denotes the horizontal lift of natural local frame field $\{\partial/\partial x^1, \cdots, \partial/\partial x^n\}$ with respect to the given Ehresmann connection θ. The set $\{dx^1, \cdots, dx^n\}$ is the dual basis of H^*. For the two bundle morphism π_* and θ from $T(TM^\times)$ onto $\widetilde{TM}$, we have

Proposition 1.6 *The bundle morphisms π_* and θ satisfy*

$$\pi_*(X^H) = X, \quad \pi_*(X^V) = 0 \tag{1.12}$$

and

$$\theta(X^H) = 0, \quad \theta(X^V) = X \tag{1.13}$$

for every $X \in A^0(\widetilde{TM})$.

1.3 Chern connection

If a Finsler metric F is given on TM, then there exists a natural metric G on $\widetilde{TM}$ defined by (1.3). Then we shall introduce a covariant derivation ∇ which satisfies some natural axioms.

For a given covariant derivation ∇ on $\widetilde{TM}$, we always define an Ehresmann connection $\theta : T(TM^\times) \to \widetilde{TM}$ by

$$\theta = \nabla \mathcal{E}. \tag{1.14}$$

With respect to the splitting (1.7), ∇ is also decomposed into the form $\nabla = \nabla^H \oplus \nabla^V$.

Definition 1.7 ([10]) The *Chern connection* on (M, F) is a covariant exterior differentiation $\nabla : A^k(\widetilde{TM}) \to A^{k+1}(\widetilde{TM})$ uniquely determined from the following conditions.

(1) ∇ is *symmetric*:

$$\nabla \pi_* = 0, \tag{1.15}$$

where we considered $\pi_* = \sum(\partial/\partial x^i) \otimes dx^i$ as a section of $A^1(\widetilde{TM})$.

(2) ∇ is *almost G-compatible*:

$$\nabla^H G = 0. \tag{1.16}$$

Remark 1.8 In the case of $C = 0$, the metric F is the norm function of a Riemannian metric g, and the Chern connection ∇ is given by $\nabla = \pi^*\nabla^M$ for the Levi-Civita connection ∇^M of (M, g). The Chern connection is also called the *Rund connection* of (M, F) (cf. [3], [12] [32]).

We can easily show that θ defined by (1.14) is invariant by the natural action $m_\bullet$ of $\mathbb{R}^+$. In local coordinate, θ is given by

$$\theta = \nabla\left(\sum \frac{\partial}{\partial x^i} \otimes y^i\right) = \sum \frac{\partial}{\partial x^i} \otimes \left(dy^i + \sum \omega^i_j y^j\right),$$

where ω^i_j is the connection forms of ∇ with respect to $\{\partial/\partial x^1, \cdots, \partial/\partial x^n\}$. The set $\{\theta^i, \cdots, \theta^n\}$ of 1-forms defined by $\theta^i = dy^i + \sum \omega^i_j y^j$ $(i = 1, \cdots, n)$ is the dual basis of V^* defined by θ.

Then the covariant derivative ∇ is also decomposed into the form $\nabla = \nabla^H \oplus \nabla^V$, where $\nabla^H : A^0(\widetilde{TM}) \longrightarrow A^0(\widetilde{TM} \otimes H^*)$ is defined by $\nabla^H_X Y = \nabla_{X^H} Y$, and $\nabla^V : A^0(\widetilde{TM}) \longrightarrow A^0(\widetilde{TM} \otimes V^*)$ is defined by $\nabla^V_X Y = \nabla_{X^V} Y$ for all $X, Y \in A^0(\widetilde{TM})$ respectively. The covariant derivative ∇G of the metric G is decomposed into the form $\nabla G = \nabla^H G + \nabla^V G$, and thus the assumption (1.16) is equivalent to $\nabla^H_X G = 0$:

$$X^H G(Y, Z) = G(\nabla^H_X Y, Z) + G(Y, \nabla^H_X Z) \tag{1.17}$$

for all $X, Y, Z \in A^0(\widetilde{TM})$. By the definition (1.4) of Cartan tensor field C, we have

$$(\nabla^V_X G)(Y, Z) = 2C(X, Y, Z) \tag{1.18}$$

for all $X, Y, Z \in A^0(\widetilde{TM})$.

On the other hand, (1.17) implies $\nabla^H_X \mathcal{E} = \theta(X^H) = 0$ and

$$X^H F^2 = X^H G(\mathcal{E}, \mathcal{E}) = G(\nabla^H_X \mathcal{E}, \mathcal{E}) + G(\mathcal{E}, \nabla^H_X \mathcal{E}) = 0.$$

Therefore we obtain

Proposition 1.9 *Let $\theta \in A^1(\widetilde{TM})$ be the Ehresmann connection of $\pi : TM^\times \to M$ defined by* (1.14) *for the Chern connection ∇ of (M, F). Then we have*

$$d^H F \equiv 0. \tag{1.19}$$

Since the condition (1.15) is equivalent to $\sum \omega^i_j \wedge dx^j = 0$, the connection form ω is given in the form $\omega^i_j = \sum \Gamma^i_{jk}(x, y) dx^k$ with the coefficients Γ^i_{jk} satisfying the symmetric property $\Gamma^i_{jk} = \Gamma^i_{kj}$. Then the condition (1.17) is written as $d^H G - {}^t\omega G - G\omega = 0$, and thus the coefficients Γ^i_{jk} are given by

$$\Gamma^i_{jk}(x, y) = \frac{1}{2} \sum G^{il} \left(\frac{\delta G_{lk}}{\delta x^j} + \frac{\delta G_{jl}}{\delta x^k} - \frac{\delta G_{jk}}{\delta x^l}\right), \tag{1.20}$$

where (G^{ij}) denotes the inverse of (G_{ij}).

1.4 Parallel translation

Let (M,F) be a Finsler manifold with the Chern connection ∇. For a non-vanishing vector field $v=\sum v^i(x)(\partial/\partial x^i)$ on M, we define its covariant derivative ∇v with respect to ∇. Let $\tilde{v}: M \to \widetilde{TM}$ be the natural lift of v defined by $\tilde{v}(x)=\mathcal{E}(v(x))=(v^*\mathcal{E})(x)$. The covariant derivative ∇v with respect to ∇ is given by $\nabla v=\tilde{v}^*\nabla\mathcal{E}=\tilde{v}^*\theta$:

$$\nabla v=\tilde{v}^*\nabla\mathcal{E}=\sum\frac{\partial}{\partial x^i}\otimes(dv^i+\sum v^j\Gamma^i_{jk}(x,v)dx^k). \tag{1.21}$$

If v satisfies $\nabla v=0$, then v is said to be *parallel* with respect to ∇.

Let $c=(x(t)):I=[0,1]\to M$ be a smooth curve, and $v(t)$ be a non-vanishing vector field along c. Then we define a lift $\tilde{c}_v: I\to TM^\times$ of c by $\tilde{c}_v=(x(t),v(t))$. A lift $\tilde{c}_v$ is said to be *horizontal* if it satisfies $\tilde{c}_v^*\theta=0$:

$$\tilde{c}_v^*\nabla\mathcal{E}=\sum\frac{\partial}{\partial x^i}\otimes\left[\frac{dv^i}{dt}+\sum v^j(t)\Gamma^i_{jk}\left(\tilde{c}_v(t)\right)\frac{dx^k}{dt}\right]=0. \tag{1.22}$$

If $v(t)$ satisfies this equation, $v(t)$ is said to be *parallel* along c.

The system (1.22) has a unique solution $v_\zeta(t)$ which depends on the initial condition $\zeta=v_\zeta(0)$ smoothly. From smooth dependence of solutions on ζ, the mapping $P_{c(t)}: T_{c(0)}M\to T_{c(t)}M$ defined by $P_{c(t)}(\zeta)=(c(t),v_\zeta(t))$ is a diffeomorphism for every $t\in I$. Because of homogeneity of θ, if $v_\zeta(t)$ is a solution of (1.22), then $\lambda v_\zeta(t)$ is also a solution satisfying $\lambda v_\zeta(0)=\lambda\zeta$, and thus the uniqueness of solutions implies that $v_{\lambda\zeta}(t)=\lambda v_\zeta(t)$. Hence the horizontal lift $\tilde{c}_v(t)$ of a curve c starting at $\zeta\in T_{c(0)}M$ satisfies the homogeneity $\tilde{c}_{\lambda v}(t)=(c(t),\lambda v_\zeta(t))=\lambda\tilde{c}_v(t)$, and $P_{c(t)}$ also satisfies the homogeneity $P_{c(t)}(\lambda\zeta)=\lambda P_{c(t)}(\zeta)$ for all $\lambda>0$ and $\zeta\in T_{c(0)}M$. The family $P_c=\{P_{c(t)}:t\in I\}$ is called the *parallel translation* along c with respect to ∇.

The tangent space T_xM at every point $x\in M$ becomes a normed linear space with a norm $\|\bullet\|_x=F(x,\bullet)$. If we put $P_{c(t)}(\zeta)=(c(t),v_\zeta(t))$ for any point ζ in $T_{c(0)}M$, the norm $\|v_\zeta(t)\|$ of the vector field $v_\zeta(t)$ along $c(t)$ is given by $F(c(t),v_\zeta(t))$. Then, because of Proposition 1.9 we have

$$dF(c(t),v_\zeta(t))=d(\tilde{c}_v^*F)=\tilde{c}_v^*(d^VF+d^HF)=\tilde{c}_v^*(d^HF)=0.$$

Hence the parallel translation P_c is norm-preserving:$\|P_{c(t)}(\zeta)\|_{c(t)}=\|\zeta\|_{c(0)}$.

Proposition 1.10 *The parallel translation P_c along any curve $c=c(t)$ on M is a norm-preserving map between the tangential normed-spaces.*

The parallel translation P_c is said to be *isometry* if it satisfies

$$\left\|P_{c(t)}(\zeta)-P_{c(t)}(\eta)\right\|_q=\|\zeta-\eta\|_p \tag{1.23}$$

for all $\zeta,\eta\in T_pM$. The parallel translation P_c along a curve $c=c(t)$ is norm-preserving, but not isometry in general. It is trivial that, if P_c is a linear mapping, then P_c is an isometry. In a later section, we shall consider the case where every tangent spaces are isometric mutually as normed linear spaces.

We denote by C_p the set of all (piecewise) smooth curves c with starting point $p=c(0)$ and ending point $p=c(1)$. Then there exists a natural product "$\circ$" in C_p. We also set