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Partial Differential Equations

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Preface

This book is intended to serve as an introduction, primarily for mathematicians, to the theory of partial differential equations and, it is hoped, should be suitable for final-year undergraduate, and first-year postgraduate, students who are following a reasonably comprehensive first course on partial differential equations. Most of the material covered in the first nine chapters has been included, at various times, in final-year courses that I have given during the past twenty years but I would be hesitant about attempting to include all this material in a single course.

The approach is classical in the sense that the methods and notation of functional analysis are not used though relatively new concepts such as 'weak solutions', 'shocks', and Green's functions which are useful to applied mathematicians are discussed. There is also no use, in any systematic fashion, of the formal theories of generalized functions or of distributions except that the elementary concept of the delta-function is freely used to develop the theory of Green's functions. My intention has been to try and emphasize the relationship between a given (generally linear) partial differential equation and the type of problems for which solutions exist and to describe the properties of solutions of the canonical second-order linear equations. I have attempted not to over-emphasize the development of special methods of solution and this aspect of the subject is largely confined to Chapter 7 where most of the general techniques for solving linear equations are described.

A number of exercises are included in the text and these vary from routine applications of the basic theory to problems taken from recent examination papers set at various Universities. I should like to thank the Universities of Cambridge, East Anglia, Liverpool, and Manchester for permission to include questions from their examination papers. Questions from Oxford University examination papers have been included by permission of Oxford University Press and I am grateful to the latter body for giving me permission to include these questions. I am also greatly indebted to my friend and colleague, Dr. R. Shail, for his help in reading, and commenting, on the complete manuscript and in reading through the proofs.

Surrey W.E.W.
November 1979

Contents

1 Introduction

THE basic definitions are given in §1.1 and the various types of problems that can occur for partial differential equations are illustrated by specific examples in §1.2. In §1.3 it is shown that the problem of determining a function which makes some integral involving it have a maximum or a minimum value is equivalent to finding a particular solution of a partial differential equation. This type of reduction is of particular relevance to mathematical physics as the behaviour of many physical systems can be formulated succinctly in terms of some maximum or minimum principle involving integrals and the methods described in §1.3 enable the governing partial differential equation to be obtained fairly easily from such maximum or minimum principles. It is important from the practical point of view that the solution of a partial differential equation varies continuously with any boundary data imposed and a problem in which this is the case is said to be 'well-posed'. This question of 'well-posedness' is considered briefly in §1.4 where a simple counter-example is used to illustrate the fact that a seemingly reasonable problem need not be well-posed. In §1.5 the basic results relating to the various Fourier series expansions of a given function are summarized, together with the corresponding results for expansion as a Fourier–Bessel series and as a series of Legendre polynomials. Particular cases of these results are required at various points in the subsequent chapters and it is convenient to collect them all together at this stage.

1.1. Basic definitions

A partial differential equation in two independent variables is a relationship of the form

$$F(x, y, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{xxx}, u_{xxy}, \ldots) = 0,$$

where $u_x = \partial u/\partial x$ etc; the *order of a partial differential equation* is the *order of the highest derivative* (*or derivatives*) occurring. This suffix notation will also be used, whenever appropriate and when no ambiguity can arise, to denote the total derivative with respect to x of a function of the single variable x.

A *linear* equation is one which is linear in the dependent variable u and all of its partial derivatives occurring in the equation. A linear equation is therefore of the form

$$Lu = g(x, y), \tag{1.1}$$

where Lu is a sum of terms each of which is a product of a function of x and y with u or one of its partial derivatives. For first- and second-order equations the respective general forms for Lu are

$$Lu = a(x, y)u_x + b(x, y)u_y + c(x, y)u,$$

$$Lu = a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u.$$

A linear equation is said to be *homogeneous* when, in equation (1.1), $g \equiv 0$ (and therefore an inhomogeneous equation corresponds to $g \not\equiv 0$). The definition of L shows that the difference between two solutions of equation (1.1) is a solution of the corresponding homogeneous equation. Thus solutions of the inhomogeneous equation can be obtained by adding a solution of the homogeneous equation to any particular solution of the inhomogeneous equation. This is analogous to writing the solution of an ordinary linear differential equation as the sum of a particular integral and a complementary function. It also follows from the definition of L that

$$L(Au_1 + Bu_2) = ALu_1 + BLu_2,$$

where A and B are constants, showing that a sum of constant multiples of solutions of the homogeneous linear equation is also a solution of the equation. This is the *principle of superposition* and it forms the basis of most practical methods of solving linear equations.

A *quasi-linear equation* is an equation which is linear in the highest derivative (or derivatives) occurring; an example of a first-order quasi-linear equation is

$$(1 + u^2)u_x + u_y = x^2.$$

An *almost linear* or *half-linear* partial differential equation is a quasi-linear equation in which the coefficients of the highest order derivatives are functions only of the independent variables; an example of such an equation is

$$x^2u_{xx}+4xyu_{yy}+uu_x+u^2=0.$$

1.2. Typical problems

Practical problems in many fields of application can be reduced to the solution of a partial differential equation or equations. Such a reduction normally requires considerable background knowledge and it is inappropriate to present here the detailed reduction for particular cases. It seems worthwhile, however, to list some typical problems which can occur in practice. The simplest such problems for second-order linear equations occur in the classical areas of physics such as heat conduction, acoustics, fluid motion, and electromagnetic theory. Historically, a detailed physical understanding of the underlying phenomena has proved invaluable in establishing the mathematical theory of various types of second-order linear partial differential equations. Such an understanding however will not be presumed and most of the results and methods will not refer to particular applications, whether physical or otherwise, as detailed knowledge of such applications is likely to be non-uniform. It should however be borne in mind that, particularly in studying a new partial differential equation, an understanding of the underlying phenomena can be extremely useful in clarifying the mathematical structure of a partial differential equation.

(i) *Simple birth process*

The probability-generating function $G(x, y)$ for a simple birth (Yule–Furry) process satisfies

$$G_y+ax(1-x)G_x=0,$$

with $G(x,0)=x^j$, where a is a constant and j a positive integer. This is a first-order linear homogeneous equation and the dependent variable is prescribed on a curve (the x-axis) in (x, y)-space.

(ii) *Incoming calls at a telephone exchange*

With certain assumptions regarding the duration of calls the problem of determining the probable number of incoming calls at a telephone exchange reduces to solving

$$G_y + a(x-1)G_x = b(x-1)G,$$

with $G(x, 0) = x^j$, where a and b are constants and j is a positive integer. This is another example of a first-order equation with the dependent variable prescribed on a curve.

(iii) *Temperature in a metallic lamina*

The temperature $u(x, y)$ in a metallic lamina, whose boundary is a closed curve C kept at a constant temperature u_0, satisfies

$$u_{xx} + u_{yy} = 0, \tag{1.2}$$

and $u = u_0$ on C, where x and y are Cartesian coordinates in the plane of the lamina. Equation (1.2) is *Laplace's equation* in two dimensions. The problem of finding the solution of a partial differential equation taking prescribed values on a closed curve (or surface in three dimensions) is termed a *Dirichlet* problem.

When C is not kept at a steady temperature but a steady known flow of heat is maintained round C then it can be shown from the theory of heat conduction that the derivative of u normal to C is known. A problem of this kind, where the normal derivative of the dependent variable is prescribed on a closed curve (or surface), is referred to as a *Neumann problem.*

If a steady source of heat such as a flame is applied within C then equation (1.2) has to be replaced by

$$u_{xx} + u_{yy} = g, \tag{1.3}$$

where g is a known function (related to the applied heat source) and this equation is the two-dimensional *Poisson's equation.*

(iv) *Transverse vibrations of a string*

The displacement u in the small transverse vibrations of an infinitely long taut string, which extends along the x-axis when in equilibrium, satisfies the equation

$$c^2 u_{xx} - u_{tt} = 0, \tag{1.4}$$

where c is a constant, and the conditions

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x), \tag{1.5}$$

where f and g are given functions. In equation (1.4), which is known as the *one-dimensional wave equation*, t is a time variable and equations (1.5) prescribe the displacement f and the velocity g at the instant $t=0$. In (x, t)-space the conditions (1.5) are applied on the line $t=0$ and can be interpreted as prescribing u and its normal derivative on this line. For a second-order equation, a problem where the dependent variable and its normal derivative are prescribed on a given curve is termed a *Cauchy problem*. The problem posed by equations (1.4) and (1.5) is, by analogy with the underlying physical problem, also referred to as an *initial-value problem*. Prescribing a function along a curve means its derivative along the curve can be found and therefore, if the normal derivative is also known, both first derivatives are known on the curve. Therefore for nth order equations a Cauchy problem is one where the dependent variable and all its derivatives of all orders up to and including the $(n-1)$th are prescribed on a curve.

For a string of finite length a and held fixed at the end points $x=0$ and $x=a$ (as, for example, for a violin string) equations (1.4) and (1.5) have to be supplemented by

$$u(0, t) = u(a, t) = 0,$$

and the resulting problem is called a mixed *initial-value–boundary-value problem*.

Equations (1.2) and (1.4) would appear to be of a similar type and yet the typical boundary conditions posed in practical problems are very dissimilar. It will transpire however that the two equations are very different in nature and that it is not possible in general to solve a Dirichlet problem for equation (1.4), and solving Cauchy problems for equation (1.2) poses serious problems. In these cases the physical context provides an excellent guide for developing an appropriate mathematical theory.

(v) *Heat conduction in a thin rod*

The temperature u in a thin straight rod with insulated sides satisfies

$$u_{xx} = ku_t, \tag{1.6}$$

where k is a constant, x is a Cartesian coordinate along the rod, and t is a time variable. A typical problem is that of determining the temperature at any time, given the temperature distribution at $t=0$ and the temperatures at the two ends $x=0$ and $x=a$, say. Thus the conditions are

$$u(0, t), u(a, t), u(x, 0) \quad \text{given.} \tag{1.7}$$

The conditions (1.7) pose a mixed *initial-value–boundary-value* problem for u similar to that for the transverse displacement of a string fixed at two ends. The main difference in the present case is that only u is prescribed at $t=0$; this is essentially because equation (1.6) only involves the first derivative with respect to t. It would not be possible to prescribe u and u_t independently as $ku_t(x, 0) = u_{xx}(x, 0)$.

It is shown in Chapter 3 that there are three different classes of second-order linear (and half-linear) equations and that equations (1.2), (1.4), and (1.6) are, respectively, typical (and in fact canonical) members of each class.

(vi) *Electrostatics*

The static electric field $\mathbf{E}$ due to a time-independent charge distribution can be shown to be of the form $\mathbf{E} = -\text{grad } u$ where

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = f, \tag{1.8}$$

and f is related to the charge density of the given distribution. Equation (1.8) is the three-dimensional *Poisson's equation* and reduces, when $f \equiv 0$, to the three-dimensional Laplace's equation. In a typical electrostatic problem u will be prescribed on some given surface and in unbounded regions will tend to zero at infinity; hence electrostatic problems are generally Dirichlet ones.

(vii) *Electromagnetic wave propagation*

Each Cartesian component of the electric and magnetic field vectors in a homogeneous medium satisfies

$$c^2 \nabla^2 u = u_{tt}, \tag{1.9}$$

where c is a constant and t is a time variable. Equation (1.9) is the *three-dimensional wave equation* and it is also satisfied by the velocity potential of small-amplitude sound waves. In physical problems u and u_t are generally known at $t=0$ at all points of space and u or $\partial u/\partial n$ are known on any bounding surface.

(viii) *Plateau's problem*

The problem of finding the surface of minimum area passing through a given plane curve C (Plateau's problem) reduces to solving

$$(1+u_y^2)u_{xx}+(1+u_x^2)u_{yy}-2u_xu_yu_{xy}=0, \tag{1.10}$$

when u is given on C.

In the following section it will be shown that the problem of solving Dirichlet and Neumann problems for some partial differential equations is equivalent to finding a function such that a given integral involving the function and its derivatives (such an integral is termed a functional) has a stationary value (often a minimum). A problem of this type is said to be a problem in *variational calculus* or a *variational problem* and use of variational calculus provides an alternative approach to the solution of partial differential equations. Many of the equations of mathematical physics can also be derived directly from some minimum principle such as one of minimum energy or Hamilton's principle of least action in mechanics.

1.3. Variational formulation of partial differential equations

The variational approach is probably best illustrated by considering a particular example before proceeding to the general case. We therefore attempt to find a function u taking specified values on the boundary C of a region D and such that the integral (or functional) $I(u)$, defined by

$$I(u)=\int_D (u_x^2+u_y^2+2gu)\,dx\,dy, \tag{1.11}$$

where g is a known function of x and y, has a minimum value. It will be assumed that u and its first and second derivatives are continuous within D. The problem is more complicated than the normal minimum problems of differential calculus as there is clearly an infinity of possible functions that could be substituted into the right-hand side of equation (1.11). The correct approach to the problem can be found by careful consideration of what is meant by stating that u produces a minimum value of I. This means that substituting into the right-hand side of equation (1.11) any function other than the correct one produces a greater value for I. In particular then

$$I(u+\varepsilon f) \geq I(u),$$

where u is the correct function and ε a constant, for any function f vanishing on C (this is necessary in order that u and $u+\varepsilon f$ take the same values on C). Hence,

$$\int_D \{u_x^2+u_y^2+2\varepsilon(u_x f_x+u_y f_y)+\varepsilon^2(f_x^2+f_y^2)+2g(u+\varepsilon f)\}\,dx\,dy \geq \int_D (u_x^2+u_y^2+2gu)\,dx\,dy$$

or

$$2\varepsilon\int_D (u_x f_x+u_y f_y+gf)\,dx\,dy+\varepsilon^2\int_D (f_x^2+f_y^2)\,dx\,dy \geq 0. \tag{1.12}$$

For small values of ε the first term on the left-hand side of equation (1.12) will dominate and hence, as ε can be positive or negative, the coefficient of ε must vanish so that

$$\int_D (u_x f_x+u_y f_y+gf)\,dx\,dy=0,$$

Hence,

$$\int_D (\text{grad}\,u\cdot\text{grad}\,f+gf)\,dx\,dy=0$$

or

$$\int_D (\text{div}\,(f\,\text{grad}\,u)-f\nabla^2 u+gf)\,dx\,dy=0. \tag{1.13}$$

Equation (1.13) can be rewritten, on using the divergence theorem, as

$$\int_C f\frac{\partial u}{\partial n}\,\mathrm{d}s - \int_D f(\nabla^2 u - g)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{1.14}$$

where $\partial u/\partial n$ is the normal derivative of u on C. The condition $f \equiv 0$ on C gives

$$\int_D f(\nabla^2 u - g)\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{1.15}$$

for all f such that $f \equiv 0$ on C.

If it is assumed that there exists a point P in D such that $\nabla^2 u - g \neq 0$ at P then, by continuity, $\nabla^2 u - g$ will be non-zero and one-signed in a neighbourhood of P. It is also possible to construct a suitable function f which is one-signed and non-zero within such a neighbourhood and vanishes outside it (e.g. $f = (a^2 - x^2)^3(b^2 - y^2)^3$ within $|x| \leq a$, $|y| \leq b$ and zero outside the rectangle satisfies all the conditions). Hence $f(\nabla^2 u - g)$ will be one-signed in some neighbourhood of P and zero outside and the left-hand side of equation (1.15) will be non-zero. This is a contradiction and hence the original assumption was false and therefore

$$\nabla^2 u = g, \tag{1.16}$$

with u known on C. Hence the problem of solving equation (1.16) with u taking prescribed values on C is equivalent to finding u taking known values on C and such that the functional I defined in equation (1.11) has a minimum value.

The above analysis can be extended to the case when I is defined by

$$I(u) = \int_D F(x, y, u, u_x, u_y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.17}$$

where F is a given function. For a minimum (or stationary) value to be attained it is necessary, by analogy with the arguments applied to equation (1.11) that the coefficient of ε in $I(u+\varepsilon f) - I(u)$ must vanish for all $f \equiv 0$ on C. Taylor's theorem gives this coefficient to be

$$\int_D (fF_u + f_x F_{u_x} + f_y F_{u_y})\,\mathrm{d}x\,\mathrm{d}y$$