V. I. Arnold

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translated from the Russian by Richard A. Silverman

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Preface

In selecting the subject matter of this book, I have attempted to confine myself to the irreducible minimum of absolutely essential material. The course is dominated by two central ideas and their ramifications: The theorem on rectifiability of a vector field (equivalent to the usual theorems on existence, uniqueness, and differentiability of solutions) and the theory of one-parameter groups of linear transformations (i.e., the theory of linear autonomous systems). Accordingly, I have taken the liberty of omitting a number of more specialized topics usually included in books on ordinary differential equations, e.g., elementary methods of integration, equations which are not solvable with respect to the derivative, singular solutions, Sturm-Liouville theory, first-order partial differential equations, etc. The last two topics are best considered in a course on partial differential equations or calculus of variations, while some of the others are more conveniently studied in the guise of exercises.

On the other hand, the applications of ordinary differential equations to mechanics are considered in more than the customary detail. Thus the pendulum equation appears at the very beginning of the book, and the efficacy of various concepts and methods introduced throughout the book are subsequently tested by applying them to this example. In this regard, the law of conservation of energy appears in the section on first integrals, the "method of small parameters" is deduced from the theorem on differentiation with respect to a parameter, and the theory of linear equations with periodic coefficients leads naturally to the study of the swing ("parametric resonance").

Many of the topics dealt with here are treated in a way drastically different from that traditionally encountered. At every point I have tried to emphasize the geometric and qualitative aspect of the phenomena under consideration. In keeping with this policy, the book is full of figures but contains no formulas of any particular complexity. On the other hand, it presents a whole congeries of fundamental concepts (like phase space and phase flows, smooth manifolds and tangent bundles, vector fields and one-parameter groups of diffeomorphisms) which remain in the shadows in the traditional coordinate-based approach. My book might have been considerably abbreviated if these concepts could have been regarded as known, but unfortunately they are not presently included in courses either on analysis or geometry. Hence I have been compelled to present them in some detail, without assuming any background on the part of the reader beyond the scope of the standard elementary courses on analysis and linear algebra.

This book stems from a year's course of lectures given by the author to students of mathematics at Moscow University during the academic years 1968–1969, and 1969–1970. In preparing the lectures for publication I have received great assistance from R. I. Bogdanov. I wish to thank him and all my colleagues and students who have commented on the preliminary mimeograph edition of the book (Moscow University, 1969). I am also grateful to D. V. Anosov and S. G. Krein for their careful reading of the manuscript.

V. I. Arnold

Frequently Used Notation

R the set (group, field) of real numbers.
C the set (group, field) of complex numbers.
Z the set (group, ring) of integers.
$\varnothing$ the empty set
$x \in X \subset Y$ an element x of a subset X of a set Y.
$X \cup Y$, $X \cap Y$ the union and intersection of the sets X and Y.
$X \setminus Y$, $X \setminus a$ the set of elements in X but not in Y, the set X minus the element $a \in X$.
$f: X \to Y$ a mapping f of a set X into a set Y.
$x \mapsto y$ the mapping carries the point x into the point y.
$f \circ g$ the product (composition) of two mappings (g is applied first).
$\exists$, $\forall$, $\Rightarrow$ there exists, for every, implies.
Theorem 0.0 the unique theorem in Sec. 0.0.
▮ end of proof symbol.
* an optional (more difficult) problem or theorem.
$\mathbf{R}^n$ a linear space of dimension n over the field **R**.
$\mathbf{R}_1 \dotplus \mathbf{R}_2$ the direct sum of the spaces $\mathbf{R}_1$ and $\mathbf{R}_2$.
$GL(\mathbf{R}^n)$ the group of linear automorphisms of $\mathbf{R}^n$.

One can consider other structures as well in the set $\mathbf{R}^n$, e.g., affine or Euclidean structure, or even the structure of the direct product of n lines. This will usually be spelled out explicitly, by referring to "the affine space $\mathbf{R}^n$," "the Euclidean space $\mathbf{R}^n$," "the coordinate space $\mathbf{R}^n$," and so on.

Elements of a linear space are called *vectors*, and are usually denoted by boldface letters ($\mathbf{v}$, $\boldsymbol{\xi}$, etc.). Vectors of the space $\mathbf{R}^n$ are identified with sets of n numbers. For example, we write $\mathbf{v} = (v_1, \ldots, v_n) = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$, where the set of n vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is called a *basis* in $\mathbf{R}^n$. The norm (length) of the vector $\mathbf{v}$ in the Euclidean space $\mathbf{R}^n$ is denoted by $|\mathbf{v}|$ and the scalar product of two vectors $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbf{R}^n$ by $(\mathbf{v}, \mathbf{w})$. Thus

$$(\mathbf{v}, \mathbf{w}) = v_1 w_1 + \cdots + v_n w_n,$$
$$|\mathbf{v}| = \sqrt{(\mathbf{v}, \mathbf{v})} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

We often deal with functions of a real parameter t called the *time*. Differentiation with respect to t (giving rise to a *velocity* or *rate of change*) is usually denoted by an overdot, as in $\dot{x} = dx/dt$.

Contents

1 Basic Concepts

1. Phase Spaces and Phase Flows

The theory of ordinary differential equations is one of the basic tools of mathematical science. The theory allows us to study all kinds of evolutionary processes with the properties of *determinacy*, *finite-dimensionality*, and *differentiability*. Before undertaking exact mathematical definitions, we consider a few examples.

1.1. Examples of evolutionary processes. A process is said to be *deterministic* if its entire future course and its entire past are uniquely determined by its state at the present instant of time. The set of all possible states of a process is called its *phase space*.

Thus, for example, classical mechanics considers the motion of systems whose past and future are uniquely determined by the initial positions and initial velocities of all points of the system. The phase space of a mechanical system is just the set whose typical element is a set of instantaneous positions and velocities of all particles of the system.

The motion of particles in quantum mechanics is not described by a deterministic process. Heat propagation is a semi-deterministic process, in that its future is determined by its present but not its past.

A process is said to be *finite-dimensional* if its phase space is finite-dimensional, i.e., if the number of parameters required to describe its state is finite. Thus, for example, the classical (Newtonian) motion of a system consisting of a finite number of particles or rigid bodies comes under this heading. In fact, the dimension of the phase space of a system of n particles is just $6n$, while that of a system of n rigid bodies is just $12n$. As examples of processes which cannot be described by using a finite-dimensional phase space, we cite the motion of fluids (studied in hydrodynamics), oscillations of strings and membranes, and the propagation of waves in optics and acoustics.

A process is said to be *differentiable* if its phase space has the structure of a differentiable manifold and if its change of state with time is described by differentiable functions. For example, the coordinates and velocities of the particles of a mechanical system vary in time in a differentiable manner, while the motions studied in shock theory do not have the differentiability property. By the same token, the motion of a system in classical mechanics can be described by using ordinary differential equations, while other tools are used in quantum mechanics, the theory of heat conduction, hydrodynamics, the theory of elasticity, optics, acoustics, and the theory of shock waves.

The process of radioactive decay and the process of reproduction of bacteria in the presence of a sufficient amount of nutrient medium afford two more examples of deterministic finite-dimensional differentiable processes. In both cases the phase space is one-dimensional, i.e., the state of the process is determined by the quantity of matter or the number of bacteria, and in both cases the process is described by an ordinary differential equation.

It should be noted that the form of the differential equation of the process and the very fact that we are dealing with a deterministic finite-dimensional differentiable process in the first place, can only be established experimentally—and hence only with a certain degree of accuracy. However, this state of affairs will not be emphasized at every turn in what follows; instead, we will talk about real processes as if they actually coincided with our idealized mathematical models.

1.2. Phase flows. An exact formulation of the general principles just presented requires the rather abstract notions of ***phase space*** and ***phase flow.*** To familiarize ourselves with these concepts, we consider an example due to N. N. Konstantinov where the simple act of introducing a phase space allows us to solve a difficult problem.

Problem 1. Two nonintersecting roads lead from City A to City B (Fig. 1). Suppose it is known that two cars connected by a rope of length less than $2l$ manage to go from A to B along different roads without breaking the rope. Can two circular wagons of radius l whose centers move along the roads in opposite directions pass each other without colliding?

Solution. Consider the square

$$M = \{(x_1, x_2) : 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

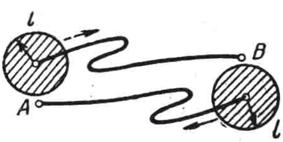

Fig. 1 Initial position of the wagons.

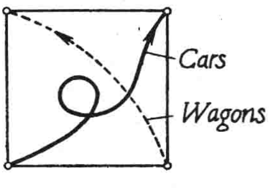

Fig. 2 Phase space of a pair of vehicles.

(Fig. 2). The position of two vehicles (one on the first road, the other on the second road) can be characterized by a point of the square M: we need only let x_i denote the fraction of the distance from A to B along the ith road which lies between A and the vehicle on the given road. Clearly there is a point of the square M corresponding to every possible state of the pair of vehicles. The square M is called the *phase space*, and its points are called *phase points*.

Thus every phase point corresponds to a definite position of the pair of vehicles (apart from their being connected), and every motion of the vehicles is represented by a motion of the phase point in the phase space. For example, the initial position of the cars (in City A) corresponds to the lower left-hand corner of the square ($x_1 = x_2 = 0$), and the motion of the cars from A to B is represented by a curve going to the opposite (upper right-hand) corner of the square. In just the same way, the initial position of the wagons corresponds to the lower right-hand corner of the square ($x_1 = 1$, $x_2 = 0$), and the motion of the wagons is represented by a curve leading to the opposite (upper left-hand) corner of the square. But every pair of curves in the square joining different pairs of opposite corners must intersect. Therefore, no matter how the wagons move, there comes a time when the pair of wagons occupies a position occupied at some time by the pair of cars. At this time the distance between the centers of the wagons will be less than $2l$, and they will not manage to pass each other.

Although differential equations play no role in the above example, the considerations which are involved closely resemble those which will concern us subsequently. Description of the states of a process as points of a suitable phase space often turns out to be extraordinarily useful.

We now return to the concepts of determinacy, finite-dimensionality, and differentiability of a process. The mathematical model of a deterministic process is a *phase flow*, which can be described as follows in intuitive terms: Let M be the phase space and $x \in M$ an initial state of a process, and let $g^t x$ denote the state of the process at time t, given that its initial state is x. For every real t this defines a mapping

$$g^t: M \to M$$

of the phase space into itself. The mapping g^t, called the *t-advance mapping*, maps every state $x \in M$ into a new state $g^t x \in M$. For example, g^0 is the identity mapping which leaves every point of M in its original position. Moreover

$$g^{t+s} = g^t g^s,$$

since the state $y = g^s x$ (Fig. 3), into which x goes after time s, goes after time

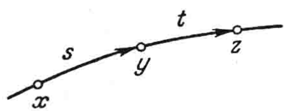

Fig. 3 Change of state of a process in the course of time.

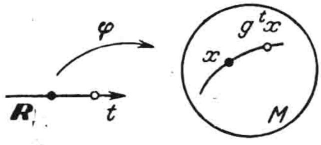

Fig. 4 Motion of a phase point in the phase space M.

t into the same state $z = g^t y$ as the state $z = g^{t+s}x$ into which x goes after time $t + s$.

Suppose we fix a phase point $x \in M$, i.e., an initial state of the process. In the course of time the state of the process will change, and the point x will describe a *phase curve* $\{g^t x, t \in \mathbf{R}\}$ in the phase space M. It is just the family of t-advance mappings $g^t\colon M \to M$ that constitutes a *phase flow*, with each phase point moving along its own phase curve.

We now turn to precise mathematical definitions. In each case M is an arbitrary set.

Definition. A family $\{g^t\}$ of mappings of a set M into itself, labelled by the set of all real numbers ($t \in \mathbf{R}$), is called a *one-parameter group of transformations* of M if

$$g^{t+s} = g^t g^s \tag{1}$$

for all s, $t \in \mathbf{R}$ and g^0 is the identity mapping (which leaves every point fixed).

Problem 2. Prove that a one-parameter group of transformations is a commutative group and that every mapping $g^t\colon M \to M$ is one-to-one.

Definition. A pair $(M, \{g^t\})$ consisting of a set M and a one-parameter group $\{g^t\}$ of transformations of M into itself is called a *phase flow*. The set M is called the *phase space* of the flow, and its elements are called *phase points*.

Definition. Let $x \in M$ be any phase point, and consider the mapping

$$\varphi\colon \mathbf{R} \to M, \qquad \varphi(t) = g^t x \tag{2}$$

of the real line into phase space (Fig. 4). Then the mapping (2) is called the *motion* of the point x under the action of the flow $(M, \{g^t\})$.

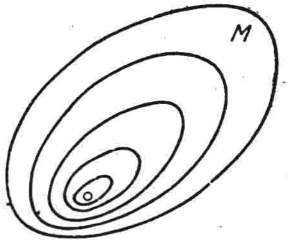

Fig. 5 Phase curves.

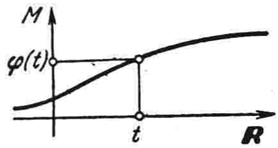

Fig. 6 An integral curve in extended phase space.

Definition. The image of $\mathbf{R}$ under the mapping (2) is called a *phase curve* of the flow $(M, \{g^t\})$. Thus a phase curve is a subset of phase space (Fig. 5).

Problem 3. Prove that there is one and only one phase curve passing through every point of phase space.

Definition. By an *equilibrium position* or *fixed point* $x \in M$ of a flow $(M, \{g^t\})$ is meant a phase point which is itself a phase curve:

$$g^t x = x \quad \forall t \in \mathbf{R}.$$

The concepts of *extended phase space* and *integral curve* are associated with the graph of the mapping φ. First we recall that the *direct product* $A \times B$ of two given sets A and B is defined as the set of all ordered pairs (a, b), $a \in A$, $b \in B$, while the *graph* of a mapping $f: A \to B$ is defined as the subset of the direct product $A \times B$ consisting of all points $(a, f(a))$, $a \in A$.

Definition. By the *extended phase space* of a flow $(M, \{g^t\})$ is meant the direct product $\mathbf{R} \times M$ of the real t-axis and the phase space M. The graph of the motion (2) is called an *integral curve* (Fig. 6) of the flow $(M, \{g^t\})$.

Problem 4. Prove that there is one and only one integral curve passing through every point of extended phase space.

Problem 5. Prove that the horizontal line $\mathbf{R} \times x$, $x \in M$ is an integral curve if and only if x is an equilibrium position.

Problem 6. Prove that a shift

$$h^s: (\mathbf{R} \times M) \to (\mathbf{R} \times M), \qquad h^s(t, x) = (t + s, x)$$

of extended phase space along the time axis carries integral curves into integral curves.

1.3. Diffeomorphisms. The above definitions formalize the concept of a deterministic process. The corresponding formalization of the concepts of finite-dimensionality and differentiability consists in requiring that the phase space be a *finite-dimensional differentiable manifold* and that the phase flow be a one-parameter group of diffeomorphisms of this manifold.

We now clarify these terms. Examples of differentiable manifolds are afforded by Euclidean spaces and their open sets, circles, spheres, tori, etc. A general definition will be given in Chap. 5, but for the time being it can be assumed that we are talking about an (open) domain of Euclidean space.

By a *differentiable function* $f: U \to \mathbf{R}$ defined in a domain U of n-dimensional Euclidean space $\mathbf{R}^n$ with coordinates $x_1, \ldots, x_n$ we mean an r-fold continuously differentiable function $f(x_1, \ldots, x_n)$ where $1 \leqslant r \leqslant \infty$. In most cases the exact value of r is of no interest and hence will not be indicated; in cases where it is required, we will allude to "r-differentiability" or the function class C^r.

By a *differentiable mapping* $f: U \to V$ of a domain U of n-dimensional Euclidean space $\mathbf{R}^n$ with coordinates $x_1, \ldots, x_n$ into a domain V of m-dimensional Euclidean space $\mathbf{R}^m$ with coordinates $y_1, \ldots, y_m$ we mean a mapping given by differentiable functions $y_i = f_i(x_1, \ldots, x_n)$. This means that if $y_i: V \to \mathbf{R}$ are the coordinates in V, then $y_i \circ f: U \to \mathbf{R}$ are differentiable functions in U $(1 \leqslant i \leqslant m)$.

By a *diffeomorphism* $f: U \to V$ we mean a one-to-one mapping such that both f and $f^{-1}: V \to U$ are differentiable mappings.

Problem 1. Which of the following functions specify a diffeomorphism $f: \mathbf{R} \to \mathbf{R}$ of the line onto the line:

$f(x) = 2x, x^2, x^3, e^x, e^x + x$?

Problem 2. Prove that if $f: U \to V$ is a diffeomorphism, then the Euclidean spaces with the domains U and V as subsets have the same dimension.
Hint. Use the implicit function theorem.

Definition. By a *one-parameter group* $\{g^t\}$ *of diffeomorphisms* of a manifold M (which can be thought of as a domain in Euclidean space) is meant a mapping

$$g: \mathbf{R} \times M \to M, \qquad g(t, x) = g^t x, \qquad t \in \mathbf{R}, \qquad x \in M$$

of the direct product $\mathbf{R} \times M$ into M such that
1) g is a differentiable mapping;
2) The mapping $g^t: M \to M$ is a diffeomorphism for every $t \in \mathbf{R}$;
3) The family $\{g^t, t \in \mathbf{R}\}$ is a one-parameter group of transformations of M.

Example 1. $M = \mathbf{R}$, $g^t x = x + vt$ $(v \in \mathbf{R})$.

Remark. Property 2 is a consequence of properties 1) and 3) (why?).

1.4. Vector fields. Let $(M, \{g^t\})$ be a phase flow, given by a one-parameter group of diffeomorphisms of a manifold M in Euclidean space.

Definition. By the *phase velocity* $\mathbf{v}(x)$ of the flow g^t at a point $x \in M$ (Fig. 7) is meant the vector representing the velocity of motion of the phase point, i.e.,

$$\left.\frac{d}{dt}\right|_{t=0} g^t x = \mathbf{v}(x). \tag{3}$$

The left-hand side of (3) is often denoted by $\dot{x}$. Note that the derivative is defined, since the motion is a differentiable mapping of a domain in Euclidean space.

Problem 1. Prove that

$$\left.\frac{d}{dt}\right|_{t=\tau} g^t x = \mathbf{v}(g^\tau x),$$

i.e., that at every instant of time the vector representing the velocity of motion of the phase point equals the vector representing the phase velocity at the very point of phase space occupied by the moving point at the given time.

Hint. See (1). The solution is given in Sec. 3.2.

If $x_1, \ldots, x_n$ are the coordinates in our Euclidean space, so that $x_i: M \to \mathbf{R}$, then the velocity vector $\mathbf{v}(x)$ is specified by n functions $v_i: M \to \mathbf{R}$, $i = 1, \ldots, n$, called the *components* of the velocity vector:

$$v_i(x) = \left.\frac{d}{dt}\right|_{t=0} x_i(g^t x).$$

Problem 2. Prove that v_i is a function of class C^{r-1} if the one-parameter group $g: \mathbf{R} \times M \to M$ is of class C^r.

Definition. Let M be a domain in Euclidean space with coordinates $x_1, \ldots, x_n$ $(x_i: M \to \mathbf{R})$, and suppose that with every point $x \in M$ there is associated the vector $\mathbf{v}(x)$ emanating from x. Then this defines a *vector field* $\mathbf{v}$ on M, specified in the x_i coordinate system by n differentiable functions $v_i: M \to \mathbf{R}$.

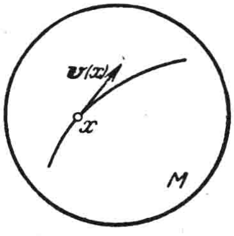

Fig. 7 The phase velocity vector.

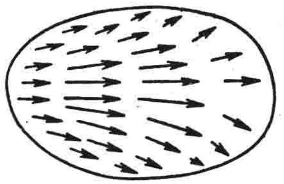

Fig. 8 A vector field.

Thus the aggregate of phase velocity vectors forms a vector field on the phase space M, namely the phase velocity field $\mathbf{v}$ (Fig. 8).

Problem 3. Prove that if x is a fixed point of a phase flow, then $\mathbf{v}(x) = 0$.

A point at which a vector of a given vector field vanishes is called a *singular point* of the vector field.† Thus the equilibrium positions of a phase flow are singular points of the phase velocity field. The converse is true, but is not so easy to prove.

1.5. The basic problem of the theory of ordinary differential equations. The basic problem of the theory of ordinary differential equations consists in investigating 1) one-parameter groups $\{g^t\}$ of diffeomorphisms of a manifold M, 2) vector fields on M, and 3) the relations between 1) and 2). We have already seen that the group $\{g^t\}$ defines a vector field on M, i.e., the field of the phase velocity $\mathbf{v}$, in accordance with formula (3). Conversely, it turns out that a vector field $\mathbf{v}$ uniquely determines a phase flow (under certain conditions to be given below).

Speaking informally, we can say that the vector field of the phase velocity gives the *local law of evolution* of a process, and that the task of the theory of ordinary differential equations is to reconstruct the past and predict the future of the process from a knowledge of this local law of evolution.

1.6. Examples of vector fields.

Example 1. It is known from experiment that *the rate of radioactive decay is proportional to the amount x of matter present at any given time.* Here the phase space is the half-line

$$M = \{x : x > 0\}$$

(Fig. 9), and the indicated experimental fact means that

$$\dot{x} = -kx, \qquad \mathbf{v}(x) = -kx, \qquad k > 0, \tag{4}$$

† Note that the components of the field have no singularities at a singular point, and in fact are continuously differentiable. The term "singular point" stems from the fact that the direction of the vectors of the field changes near such a point, in general discontinuously.

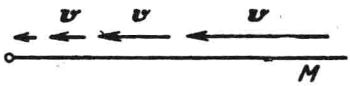

Fig. 9 The phase space of radioactive decay.

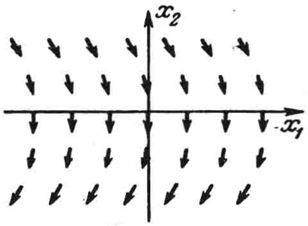

Fig. 10 The phase plane for vertical fall.

i.e., the vector field **v** on the half line is directed toward 0 and the magnitude of the phase velocity vector is proportional to x.

Example 2. It is known from experiment that *the reproduction rate of a colony of bacteria supplied with enough food is proportional to the quantity x of bacteria present at any given time.* Again M is the half-line $x > 0$, but the vector field differs in sign from that of the previous example:

$$\dot{x} = kx, \qquad \mathbf{v}(x) = kx, \qquad k > 0. \tag{5}$$

Note that equation (5) corresponds to growth, with the increase proportional to the number of individuals present.

Example 3. One can imagine a situation where *the increase is proportional to the total number of pairs present*, i.e.,

$$\dot{x} = kx^2, \qquad \mathbf{v}(x) = kx^2 \tag{6}$$

(this situation is more readily encountered in physical chemistry than in biology). Later we will see the catastrophic consequences of the excessively rapid law of growth (6).

Example 4. Vertical fall of a particle to the ground (starting from not too great an initial height) is described experimentally by Galileo's law, which asserts that the acceleration is constant. Here the phase space M is the plane (x_1, x_2), where x_1 is the height and x_2 the velocity, while Galileo's law is expressed by formulas like (3), namely

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g \tag{7}$$

($-g$ is the acceleration due to gravity). The corresponding vector field of the phase velocity has components $v_1 = x_2$, $v_2 = -g$ (Fig. 10).