ALM 30

Advanced Lectures in Mathematics

Automorphic Forms and *L*-functions

自守形式与*L*-函数

Editor: Jianya Liu

HIGHER EDUCATION PRESS

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ADVANCED LECTURES IN MATHEMATICS

The CIMPA-UNESCO-CHINA Research School 2010: Automorphic Forms and L-functions

Preface

This is a collection of lecture notes from the CIMPA-UNESCO-CHINA Research School 2010: Automorphic Forms and L-functions, held at Shandong University at Weihai, in August 1–14, 2010.

The scientific directors were W. Duke and P. Sarnak (Chair) who provided detailed advice regarding the important aspects of the research school. The invited speakers were J. Cogdell, G. Harcos, E. Lapid, Xiaoqing Li, Wenzhi Luo, P. Michel, A. Reznikov, F. Shahidi, and Yangbo Ye.

Each of the speakers was scheduled to give a series of four lectures. Shahidi, although could not come to Weihai in person, has given full support to the research school by making his notes available, and Cogdell expertly combined his own lecture notes with Shahidi's material to give eight lectures. Not only have the speakers given excellent lectures, but they have also spent time completing their lecture notes – available for us here. Besides the speakers, there were fifteen scholars from all over the world plus ten international graduate students and over sixty Chinese graduate students who participated in various activities offered by the research school. The diligence and enthusiasm, professional skills and hard work of speakers, participants and organizers have contributed to the great success of the research school.

M. Waldschmidt visited Weihai in 2007 and provided clear instructions on how to organize a CIMPA school. M. Jambu, representative of CIMPA, not only participated the whole school, but also conducted lots of administrative duties on behalf of CIMPA before and after the school.

It is a great pleasure to express my sincere thanks to all the friends who have made this research school possible and who have contributed to its success. Thanks are also due to Lizhen Ji, Liping Wang, and Huaying Li for their effort and patience, and to Taiyu Li for helping me prepare the TeX file for publication. This event would not have been possible without the financial support from Shandong University, the NSFC, and CIMPA-UNESCO, and I would like to express my gratitude and thanks to them.

Jianya Liu
School Coordinator
July 2013

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L-functions and Functoriality

James W. Cogdell*

Abstract

The principle of functoriality is one of the central tenets of the Langlands program. There are two main approaches to functoriality, via the trace formula or via the method of L-functions. In this article we discuss the method of L-functions, pioneered by Piatetski-Shapiro. This method primarily deals with functoriality in the case where the target group is GL_n, the fundamental tool here being the converse theorem for GL_n. As a vehicle for functoriality, the input to the converse theorem must be checked, and this is done by controlling the L-functions of the automorphic representations to be transferred. For the cases presented here these L-functions are controlled by the Langlands-Shahidi method, which we outline as well. We discuss most of the major cases of functoriality that have been obtained by combining the converse theorem for GL_n and the Langlands-Shahidi method.

2010 Mathematics Subject Classification: 11F70, 11R39, 22E50, 22E55.

Keywords and Phrases: Automorphic forms, L-functions, Converse theorem, Eisenstein series, Langlands-Shahidi method, Functoriality.

The principle of functoriality is one of the central tenets of the Langlands program. It is a purely automorphic avatar of Langlands' vision of a non-abelian class field theory. One can find an outline of this in Section 5 below. There are two main approaches to functoriality. The one envisioned by Langlands is through the Arthur-Selberg trace formula. With the recent proof of the Fundamental Lemma by Ngô, Waldspurger, and others this method is now available and will be the subject of a forthcoming book of Jim Arthur [1]. The second method is that of L-functions.

The method of L-functions was pioneered by Piatetski-Shapiro. It primarily deals with functoriality in the case where the target group is GL_n. The fundamental tool here is the converse theorem for GL_n, as explained in Section 6 below. As

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Piatetski-Shapiro said "Arthur's approach is more general, but this approach is much easier." (see Shahidi's contribution to [6]). The converse theorem is a way to tell when a representation of $GL_n(\mathbb{A})$ is automorphic based on the analytic properties of its L-functions (see Section 2 below). As a vehicle for functoriality, the input to the converse theorem must be checked, and this is done by controlling the L-functions of the automorphic representations to be functorially transferred. There are two ways to control these L-functions: via integral representations and via Eisenstein series, or the Langlands-Shahidi method. For the examples discussed here, these are controlled by the Langlands-Shahidi method, which we discuss in Sections 3–5. The functorial liftings themselves are obtained in Sections 7 and 8.

While Piatetski-Shapiro viewed this method as being simpler, it is in many ways more flexible and still mysterious. For example, as presented in Section 8, by this method you can obtain the third and fourth symmetric power lifting from GL_2 and these are still not attainable by the trace formula. In general, whenever you can control enough twisted L-functions, you can apply the converse theorem and obtain functoriality to GL_n. Of course controlling these L-functions is hard and for most of them we do not have a way to analyze them: they either fall outside the range of the Langlands-Shahidi method or we do not yet know if we can find integral representations for them. So there is still much work to be done.

These notes represent the lectures I gave at the CIMPA-UNESCO-CHINA Research School on *Automorphic Forms and L-functions* in August of 2010. Their purpose was to present an introduction to the L-function approach to functoriality. My original lectures were to be on GL_n and functoriality. These comprise Sections 1 and 2 here, which present the theory of automorphic representations and L-functions for GL_n, including the converse theorem, and then Sections 6 and 7 which were devoted to an exposition of functoriality via L-functions and then the example of functoriality for the classical groups. However, Shahidi was not able to attend the conference at the last moment, so I also gave an informal introduction to the Langlands-Shahidi method, based on Shahidi's notes [10]. These appear as Sections 3–5 and 8 here. These sections can be viewed as a "gentle introduction" to [10] and I have tried to indicate where the results here can be found in Shahidi's contribution. However, having the opportunity to give a single self-contained introduction to our approach to functoriality by the method of L-functions, I took the opportunity to integrate both series of lecture into a single contribution. I hope the reader finds this useful.

A word on references. I have surveyed the material in Sections 1,2,6 and 7 many times. Rather than burden this set of informal notes with pages of references, I refer instead to the sources I used for these talks, which are mainly my previous surveys. One can find more extensive references there. Similarly, for Sections 3–5 and 8 I have included in the bibliography the sources that I have used, which were surveys by Kim and Shahidi, Shahidi's new book, and Shahidi's contribution to this volume. I hope the reader does not mind this informality.

Finally I would like to thank Jianya Liu, the faculty at Shandong University, and all the students that attended for giving me the opportunity for presenting these lectures.

I L-functions for $GL(n)$ and Converse Theorems

1 Modular Forms and Automorphic Representations

1.1 Classical Modular Forms and Their L-functions

We begin our tale by recalling the classical results of Hecke and Weil.

Let $\mathfrak{H} = \{z = x + iy \mid y > 0\}$ denote the complex upper half-plane. Then a (classical) modular form of weight k for $\Gamma = SL_2(\mathbb{Z})$ is a function $f : \mathfrak{H} \to \mathbb{C}$ which is holomorphic on $\mathfrak{H}$ and at the cusps of Γ and for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

We denote the space of these by $M_k(\Gamma)$. These arise in the theory of theta series, elliptic modular forms, etc. Since $f(z+1) = f(z)$, by modularity under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this will have a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

and is called a *cusp form* if

$$a_0 = \int_0^1 f(x+iy)\, dx = 0.$$

Recall that $\Gamma\backslash\mathfrak{H}$ is the Riemann sphere with the "cusp at ∞" removed, so cusp forms are those modular forms that vanish at the cusps. The subspace of cusp forms of weight k is denoted by $S_k(\Gamma)$.

To each cusp form Hecke associated an L-function given by the Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent for $\mathrm{Re}(s) > \frac{k}{2} + 1$. The L-function is related to f through a Mellin transform:

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^\infty f(iy) y^s\, d^\times y.$$

By transferring the analytic properties of f to $L(s,f)$, Hecke showed:

Theorem 1.1. *If $f \in S_k(\Gamma)$ then $\Lambda(s,f)$ is nice, i.e.,*

(1) *$\Lambda(s,f)$ extends to an entire function of s;*

(2) *$\Lambda(s,f)$ is bounded in vertical strips (BVS);*

(3) *$\Lambda(s,f)$ satisfied the functional equation (FE)*

$$\Lambda(s,f) = (i)^k \Lambda(k-s,f).$$

Note that the functional equation results from the modular transformation law of $f(z)$ under $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, namely $f(Sz) = f\left(\frac{-1}{z}\right) = z^k f(z)$.

Remark 1.1. In general $L(s,f)$ need not have an *Euler product.* Hecke introduced an algebra of operators $\mathcal{H} = \langle T_p \mid p \text{ prime}\rangle$, the original Hecke algebra, such that if f is a simultaneous eigen-function for all T_n, i.e., $T_n f = \lambda_n f$, and if we normalize f such that $a_1 = 1$ then $\lambda_n = a_n$ and we have

$$L(s,f) = \prod_p (1 - a_p p^{-s} + p^{2k-1} p^{-2s})^{-1}.$$

Hecke was able to invert this process, via the inverse Mellin transform, and prove the "Hecke Converse Theorem".

Theorem 1.2. *Let*

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series that converges in some right half-plane. Set

$$\Lambda(s) = (2\pi)^{-s}\Gamma(s)D(s).$$

If $\Lambda(s)$ is nice *as above, i.e.,* (1) *entire continuation,* (2) *BVS, and* (3) *satisfies the functional equation $\Lambda(s) = (i)^k \Lambda(k-s)$, then*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is a cusp form of weight k for $\Gamma = SL_2(\mathbb{Z})$.

Note that by the Fourier expansion $f(z+1) = f(z)$, so $f(z)$ is modular under the translation matrix T. Since $SL_2(\mathbb{Z})$ is generated by T and the inversion S, we only need the transformation law under S. This follows from the functional equation for $\Lambda(s)$ via Mellin inversion

$$f(iy) = \sum_{n=1}^{\infty} a_n e^{-2\pi n y} = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=2} \Lambda(s) y^{-s}\, ds.$$

If f is a cusp form not for all of $SL_2(\mathbb{Z})$ but say for the Hecke congruence group of level N

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{N} \right\}$$

then the inversion S is no longer in Γ. Weil still defined $L(s, f)$ for $f \in S_k(\Gamma)$ and proved they were nice, using now $S_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, which normalizes $\Gamma_0(N)$, in place of S. (Now the functional equation relates f and a form g related to f through S_N.) To invert this process, i.e., to establish the "Weil Converse Theorem", Weil showed that besides knowing that $D(s)$ was nice, one had to also control twisted Dirichlet series; one needed that the

$$D_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n) a_n}{n^s}$$

were *nice* (essentially) for all Dirichlet characters χ of conductor relatively prime to N. The important thing to note here is that one Dirichlet series no longer suffices for inversion, one must control a family of twisted Dirichlet series as well.

1.2 Automorphic Representations of $GL(n)$

The modern theory of automorphic representations is a theory of functions on adele groups. To make the connection note that

$$\mathfrak{H} = PGL_2^+(\mathbb{R})/PSO_2(\mathbb{R}).$$

So functions of $\mathfrak{H}$ can be lifted to functions on $GL_2(\mathbb{R})$. So we now have functions on a group.

To make an adelic theory, take k a number field ... but for now we can take $k = \mathbb{Q}$. The the adele ring of $\mathbb{Q}$ is

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \varinjlim_S (\mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p) = {\prod_v}' \mathbb{Q}_v,$$

where the limit is over all finite subsets S of primes of $\mathbb{Z}$. Then $\mathbb{Q} \hookrightarrow \mathbb{A}$ as a discrete, co-compact subgroup with $\mathbb{Q} \cap (\mathbb{R} \times \prod_p \mathbb{Z}_p) = \mathbb{Z}$, so that $\mathbb{Z}\backslash\mathbb{R} \simeq \mathbb{Q}\backslash\mathbb{A}/\prod_p \mathbb{Z}_p$.

Similarly, for $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$,

$$GL_n(\mathbb{A}) = \varinjlim_S (GL_n(\mathbb{R}) \times \prod_{p \in S} GL_n(\mathbb{Q}_p) \times \prod_{p \notin S} GL_n(\mathbb{Z}_p)) = {\prod_v}' GL_n(\mathbb{Q}_v)$$

and $GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A})$ as a canonical discrete subgroup. While the quotient $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})$ is no longer compact, it is finite volume modulo the center $Z_n(\mathbb{A}) \simeq \mathbb{A}^\times$.

The analogue of the space of classical modular forms is the space of automorphic forms

$$\mathcal{A}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}); \omega)$$

consisting of functions $\varphi : GL_n(\mathbb{A}) \to \mathbb{C}$ such that $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in GL_n(\mathbb{Q})$ plus regularity and growth conditions to match the classical conditions of holomorphy and holomorphy at the cusps. Here ω is a fixed unitary central character, i.e., φ satisfies $\varphi(zg) = \omega(z)\varphi(g)$ for $z \in Z_n(\mathbb{A})$. $GL_n(\mathbb{A})$ acts on the