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（影印版）80

Anders Vretblad

Fourier Analysis and Its Applications

傅里叶分析及其应用

科学出版社

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《国外数学名著系列》(影印版) 序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Preface

The classical theory of Fourier series and integrals, as well as Laplace transforms, is of great importance for physical and technical applications, and its mathematical beauty makes it an interesting study for pure mathematicians as well. I have taught courses on these subjects for decades to civil engineering students, and also mathematics majors, and the present volume can be regarded as my collected experiences from this work.

There is, of course, an unsurpassable book on Fourier analysis, the treatise by Katznelson from 1970. That book is, however, aimed at mathematically very mature students and can hardly be used in engineering courses. On the other end of the scale, there are a number of more-or-less cookbook-styled books, where the emphasis is almost entirely on applications. I have felt the need for an alternative in between these extremes: a text for the ambitious and interested student, who on the other hand does not aspire to become an expert in the field. There do exist a few texts that fulfill these requirements (see the literature list at the end of the book), but they do not include all the topics I like to cover in my courses, such as Laplace transforms and the simplest facts about distributions.

The reader is assumed to have studied real calculus and linear algebra and to be familiar with complex numbers and uniform convergence. On the other hand, we do not require the Lebesgue integral. Of course, this somewhat restricts the scope of some of the results proved in the text, but the reader who *does* master Lebesgue integrals can probably extrapolate the theorems. Our ambition has been to prove as much as possible within these restrictions.

Some knowledge of the simplest distributions, such as point masses and dipoles, is essential for applications. I have chosen to approach this matter in two separate ways: first, in an intuitive way that may be sufficient for engineering students, in star-marked sections of Chapter 2 and subsequent chapters; secondly, in a more strict way, in Chapter 8, where at least the fundaments are given in a mathematically correct way. Only the one-dimensional case is treated. This is not intended to be more than the merest introduction, to whet the reader's appetite.

Acknowledgements. In my work I have, of course, been inspired by existing literature. In particular, I want to mention a book by Arne Broman, *Introduction to Partial Differential Equations...* (Addison–Wesley, 1970), a compendium by Jan Petersson of the Chalmers Institute of Technology in Gothenburg, and also a compendium from the Royal Institute of Technology in Stockholm, by Jockum Aniansson, Michael Benedicks, and Karim Daho. I am grateful to my colleagues and friends in Uppsala. First of all Professor Yngve Domar, who has been my teacher and mentor, and who introduced me to the field. The book is dedicated to him. I am also particularly indebted to Gunnar Berg, Christer O. Kiselman, Anders Källström, Lars-Åke Lindahl, and Lennart Salling. Bengt Carlsson has helped with ideas for the applications to control theory. The problems have been worked and re-worked by Jonas Bjermo and Daniel Domert. If any incorrect answers still remain, the blame is mine.

Finally, special thanks go to three former students at Uppsala University, Mikael Nilsson, Matthias Palmér, and Magnus Sandberg. They used an early version of the text and presented me with very constructive criticism. This actually prompted me to pursue my work on the text, and to translate it into English.

Uppsala, Sweden
January 2003

Anders Vretblad

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1

Introduction

1.1 The classical partial differential equations

In this introductory chapter, we give a brief survey of three main types of partial differential equations that occur in classical physics. We begin by establishing some convenient notation.

Let Ω be a domain (an open and connected set) in three-dimensional space $\mathbf{R}^3$, and let T be an open interval on the time axis. By $C^k(\Omega)$, resp. $C^k(\Omega \times T)$, we mean the set of all real-valued functions $u(x, y, z)$, resp. $u(x, y, z, t)$, with all their partial derivatives of order up to and including k defined and continuous in the respective regions. It is often practical to collect the three spatial coordinates (x, y, z) in a vector $\mathbf{x}$ and describe the functions as $u(\mathbf{x})$, resp. $u(\mathbf{x}, t)$. By Δ we mean the LAPLACE operator

$$\Delta = \nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

Partial derivatives will mostly be indicated by subscripts, e.g.,

$$u_t = \frac{\partial u}{\partial t}, \qquad u_{yx} = \frac{\partial^2 u}{\partial x \partial y} .$$

The first equation to be considered is called the *heat equation* or the *diffusion equation:*

$$\Delta u = \frac{1}{a^2} \frac{\partial u}{\partial t}, \qquad (\mathbf{x}, t) \in \Omega \times T.$$

As the name indicates, this equation describes conduction of heat in a homogeneous medium. The temperature at the point $\mathbf{x}$ at time t is given by $u(\mathbf{x}, t)$, and a is a constant that depends on the conducting properties of the medium. The equation can also be used to describe various processes of diffusion, e.g., the diffusion of a dissolved substance in the solvent liquid, neutrons in a nuclear reactor, BROWNian motion, etc.

The equation represents a category of second-order partial differential equations that is traditionally categorized as *parabolic*. Characteristically, these equations describe *non-reversible* processes, and their solutions are highly regular functions (of class C^∞).

In this book, we shall solve some special problems for the heat equation. We shall be dealing with situations where the spatial variable can be regarded as one-dimensional: heat conduction in a homogeneous rod, completely isolated from the exterior (except possibly at the ends of the rod). In this case, the equation reduces to

$$u_{xx} = \frac{1}{a^2} u_t .$$

The *wave equation* has the form

$$\Delta u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}, \qquad (\mathbf{x}, t) \in \Omega \times T.$$

where c is a constant. This equation describes vibrations in a homogeneous medium. The value $u(\mathbf{x}, t)$ is interpreted as the deviation at time t from the position at rest of the point with rest position given by $\mathbf{x}$.

The equation is a case of *hyperbolic* equations. Equations of this category typically describe reversible processes (the past can be deduced from the present and future by "reversion of time"). Sometimes it is even suitable to allow solutions for which the partial derivatives involved in the equation do not exist in the usual sense. (Think of shock waves such as the sonic bangs that occur when an aeroplane goes supersonic.) We shall be studying the *one-dimensional* wave equation later on in the book. This case can, for instance, describe the motion of a vibrating string.

Finally we consider an equation that does not involve time. It is called the *Laplace* equation and it looks simply like this:

$$\Delta u = 0.$$

It occurs in a number of physical situations: as a special case of the heat equation, when one considers a stationary situation, a *steady state*, that does not depend on time (so that $u_t = 0$); as an equation satisfied by the potential of a conservative force; and as an object of considerable purely mathematical interest. Together with the closely related POISSON equation, $\Delta u(\mathbf{x}) = F(\mathbf{x})$, where F is a known function, it is typical of equations classified as *elliptic*. The solutions of the Laplace equation are very regular functions: not only do they have derivatives of all orders, there are even certain possibilities to reconstruct the whole function from its local behaviour near a single point. (If the reader is familiar with analytic functions, this should come as no news in the two-dimensional case: then the solutions are harmonic functions that can be interpreted (locally) as real parts of analytic functions.)

The names *elliptic, parabolic,* and *hyperbolic* are due to superficial similarities in the appearance of the differential equations and the equations of conics in the plane. The precise definitions of the different types are as follows: The unknown function is $u = u(\mathbf{x}) = u(x_1, x_2, \ldots, x_m)$. The equations considered are *linear;* i.e., they can be written as a sum of terms equal to a known function (which can be identically zero), where each term in the sum consists of a coefficient (constant or variable) times some derivative of u, or u itself. The derivatives are of degree at most 2. By changing variables (possibly locally around each point in the domain), one can then write the equation so that no mixed derivatives occur (this is analogous to the diagonalization of quadratic forms). It then reduces to the form

$$a_1u_{11} + a_2u_{22} + \cdots + a_mu_{mm} + \{\text{terms containing } u_j \text{ and } u\} = f(\mathbf{x}),$$

where $u_j = \partial u/\partial x_j$ etc. If all the a_j have the same sign, the equation is elliptic; if at least one of them is zero, the equation is parabolic; and if there exist a_j's of opposite signs, it is hyperbolic.

An equation can belong to different categories in different parts of the domain, as, for example, the TRICOMI equation $u_{xx} + xu_{yy} = 0$ (where $u = u(x, y)$), which is elliptic in the right-hand half-plane and hyperbolic in the left-hand half-plane. Another example occurs in the study of the so-called velocity potential $u(x, y)$ for planar laminary fluid flow. Consider, for instance, an aeroplane wing in a streaming medium. In the case of *ideal* flow one has $\Delta u = 0$. Otherwise, when there is friction (air resistance), the equation looks something like $(1-M^2)u_{xx}+u_{yy} = 0$, with $M = v/v_0$, where v is the speed of the flowing medium and v_0 is the velocity of sound in the medium. This equation is elliptic, with nice solutions, as long as $v < v_0$, while it is hyperbolic if $v > v_0$ and then has solutions that represent shock waves (sonic bangs). Something quite complicated happens when the speed of sound is surpassed.

1.2 Well-posed problems

A *problem* for a differential equation consists of the equation together with some further conditions such as initial or boundary conditions of some form. In order that a problem be "nice" to handle it is often desirable that it have certain properties:

1. There *exists* a solution to the problem.
2. There exists *only one* solution (i.e., the solution is uniquely determined).
3. The solution is *stable*, i.e., small changes in the given data give rise to small changes in the appearance of the solution.

A problem having these properties (the third condition must be made precise in some way or other) is traditionally said to be *well posed*. It is, however, far from true that all physically relevant problems are well posed. The third condition, in particular, has caught the attention of mathematicians in recent years, since it has become apparent that it is often very hard to satisfy it. The study of these matters is part of what is popularly labeled chaos research.

To satisfy the reader's curiosity, we shall give some examples to illuminate the concept of well-posedness.

Example 1.1. It can be shown that for suitably chosen functions $f \in C^\infty$, the equation $u_x + u_y + (x + 2iy)u_t = f$ has no solution $u = u(x, y, t)$ at all (in the class of complex-valued functions) (Hans Lewy, 1957). Thus, in this case, condition 1 fails. □

Example 1.2. A natural problem for the heat equation (in one spatial dimension) is this one:

$$u_{xx}(x,t) = u_t(x,t),\ x > 0,\ t > 0; \quad u(x,0) = 0,\ x > 0; \quad u(0,t) = 0,\ t > 0.$$

This is a mathematical model for the temperature in a semi-infinite rod, represented by the positive x-axis, in the situation when at time 0 the rod is at temperature 0, and the end point $x = 0$ is kept at temperature 0 the whole time $t > 0$. The obvious and intuitive solution is, of course, that the rod will remain at temperature 0, i.e., $u(x,t) = 0$ for all $x > 0$, $t > 0$. But the mathematical problem has additional solutions: let

$$u(x,t) = \frac{x}{t^{3/2}}\, e^{-x^2/(4t)}, \quad x > 0,\ t > 0.$$

It is a simple exercise in partial differentiation to show that this function satisfies the heat equation; it is obvious that $u(0,t) = 0$, and it is an easy exercise in limits to check that $\lim_{t \searrow 0} u(x,t) = 0$. The function must be considered a solution of the problem, as the formulation stands. Thus, the problem fails to have property 2.

The disturbing solution has a rather peculiar feature: it could be said to represent a certain (finite) amount of heat, located at the end point of the rod at time 0. The value of $u(\sqrt{2t}, t)$ is $\sqrt{(2/e)}/t$, which tends to $+\infty$ as $t \searrow 0$. One way of excluding it as a solution is adding some condition to the formulation of the problem; as an example it is actually sufficient to demand that a solution must be bounded. (We do not prove here that this does solve the dilemma.) □

Example 1.3. A simple example of instability is exhibited by an ordinary differential equation such as $y''(t) + y(t) = f(t)$ with initial conditions $y(0) = 1$, $y'(0) = 0$. If, for example, we take $f(t) = 1$, the solution is $y(t) = 1$. If we introduce a small perturbation in the right-hand member by taking $f(t) = 1 + \varepsilon \cos t$, where $\varepsilon \neq 0$, the solution is given by $y(t) = 1 + \frac{1}{2}\varepsilon t \sin t$. As time goes by, this expression will oscillate with increasing amplitude and "explode". The phenomenon is called *resonance*. □

1.3 The one-dimensional wave equation

We shall attempt to find *all* solutions of class C^2 of the one-dimensional wave equation

$$c^2 u_{xx} = u_{tt}.$$

Initially, we consider solutions defined in the open half-plane $t > 0$.

Introduce new coordinates (ξ, η), defined by

$$\xi = x - ct, \qquad \eta = x + ct.$$

It is an easy exercise in applying the chain rule to show that

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$
$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right).$$

Inserting these expressions in the equation and simplifying we obtain

$$c^2 \cdot 4\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0 \quad \iff \quad \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0.$$

Now we can integrate step by step. First we see that $\partial u/\partial \eta$ must be a function of only η, say, $\partial u/\partial \eta = h(\eta)$. If ψ is an antiderivative of h, another integration yields $u = \varphi(\xi) + \psi(\eta)$, where φ is a new arbitrary function. Returning to the original variables (x, t), we have found that

$$u(x,t) = \varphi(x - ct) + \psi(x + ct). \tag{1.1}$$

In this expression, φ and ψ are more-or-less arbitrary functions of one variable. If the solution u really is supposed to be of class C^2, we must demand that φ and ψ have continuous second derivatives.

It is illuminating to take a closer look at the significance of the two terms in the solution. First, assume that $\psi(s) = 0$ for all s, so that $u(x,t) =$

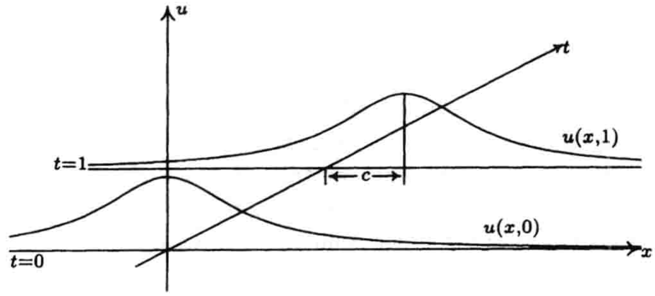

FIGURE 1.1.

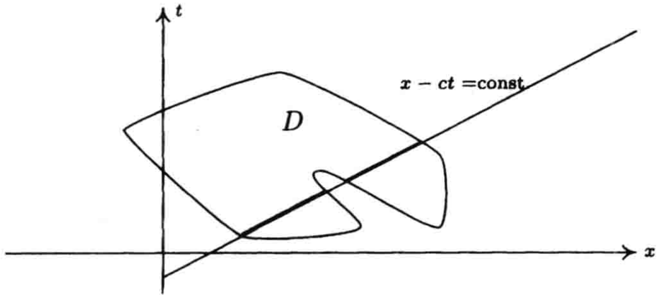

FIGURE 1.2.

$\varphi(x - ct)$. For $t = 0$, the graph of the function $x \mapsto u(x, 0)$ looks just like the graph of φ itself. At a later moment, the graph of $x \mapsto u(x, t)$ will have the same shape as that of φ, but it is pushed ct units of length to the right. Thus, the term $\varphi(x - ct)$ represents a *wave moving to the right along the x-axis* with constant speed equal to c. See Figure 1.1! In an analogous manner, the term $\psi(x + ct)$ describes a wave moving to the left with the same speed. The general solution of the one-dimensional wave equation thus consists of a superposition of two waves, moving along the x-axis in opposite directions.

The lines $x \pm ct$ = constant, passing through the half-plane $t > 0$, constitute a net of level curves for the two terms in the solution. These lines are called the *characteristic curves* or simply *characteristics* of the equation. If, instead of the half-plane, we study solutions in some other region D, the derivation of the general solution works in the same way as above, as long as the characteristics run unbroken through D. In a region such as that shown in Figure 1.2, the function φ need not take on the same value on the two indicated sections that do lie on the same line but are not connected inside D. In such a case, the general solution must be described in a more complicated way. But if the region is *convex*, the formula (1.1) gives the general solution.

Remark. In a way, the general behavior of the solution is similar also in higher spatial dimensions. For example, the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

has solutions that represent wave-shapes passing the plane in all directions, and the general solution can be seen as a sort of superposition of such solutions. But here the directions are infinite in number, and there are both planar and circular wave-fronts to consider. The superposition cannot be realized as a sum — one has to use integrals. It is, however, usually of little interest to exhibit the general solution of the equation. It is much more valuable to be able to pick out some particular solution that is of importance for a concrete situation. □

Let us now solve a natural *initial value problem* for the wave equation in one spatial dimension. Let $f(x)$ and $g(x)$ be given functions on $\mathbf{R}$. We want to find all functions $u(x,t)$ that satisfy

$$\text{(P)} \qquad \begin{cases} c^2 u_{xx} = u_{tt}\,, & -\infty < x < \infty, \quad t > 0; \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & -\infty < x < \infty. \end{cases}$$

(The initial conditions assert that we know the shape of the solution at $t = 0$, and also its rate of change at the same time.) By our previous calculations, we know that the solution must have the form (1.1), and so our task is to determine the functions φ and ψ so that

$$f(x) = u(x,0) = \varphi(x) + \psi(x), \quad g(x) = u_t(x,0) = -c\,\varphi'(x) + c\,\psi'(x). \tag{1.2}$$

An antiderivative of g is given by $G(x) = \int_0^x g(y)\,dy$, and the second formula can then be integrated to

$$-\varphi(x) + \psi(x) = \frac{1}{c}\,G(x) + K,$$

where K is the integration constant. Combining this with the first formula of (1.2), we can solve for φ and ψ:

$$\varphi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\,G(x) - K\right), \qquad \psi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\,G(x) + K\right).$$

Substitution now gives

$$\begin{aligned} u(x,t) &= \varphi(x - ct) + \psi(x + ct) \\ &= \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\,G(x-ct) - K + f(x+ct) + \frac{1}{c}\,G(x+ct) + K\right) \\ &= \frac{f(x-ct) + f(x+ct)}{2} + \frac{G(x+ct) - G(x-ct)}{2c} \\ &= \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy. \end{aligned} \tag{1.3}$$

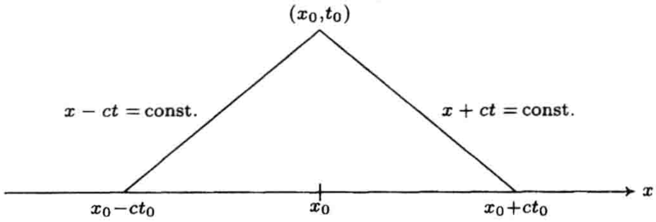

FIGURE 1.3.

The final result is called D'ALEMBERT'S formula. It is something as rare as an explicit (and unique) solution of a problem for a partial differential equation.

Remark. If we want to compute the value of the solution $u(x,t)$ at a particular point (x_0, t_0), d'Alembert's formula tells us that it is sufficient to know the initial values on the interval $[x_0 - ct_0, x_0 + ct_0]$: this is again a manifestation of the fact that the "waves" propagate with speed c. Conversely, the initial values taken on $[x_0 - ct_0, x_0 + ct_0]$ are sufficient to determine the solution in the isosceles triangle with base equal to this interval and having its other sides along characteristics. See Figure 1.3. □

In a similar way one can solve suitably formulated problems in other regions. We give an example for a semi-infinite spatial interval.

Example 1.4. Find all solutions $u(x,t)$ of $u_{xx} = u_{tt}$ for $x > 0$, $t > 0$, that satisfy $u(x,0) = 2x$ and $u_t(x,0) = 1$ for $x > 0$ and, in addition, $u(0,t) = 2t$ for $t > 0$.

Solution. Since the first quadrant of the xt-plane is convex, all solutions of the equation must have the appearance

$$u(x,t) = \varphi(x-t) + \psi(x+t), \qquad x > 0,\ t > 0.$$

Our task is to determine what the functions φ and ψ look like. We need information about $\psi(s)$ when s is a positive number, and we must find out what $\varphi(s)$ is for all real s.

If $t = 0$ we get $2x = u(x,0) = \varphi(x) + \psi(x)$ and $1 = u_t(x,0) = -\varphi'(x) + \psi'(x)$; and for $x = 0$ we must have $2t = \varphi(-t) + \psi(t)$. To liberate ourselves from the magic of letters, we neutralize the name of the variable and call it s. The three conditions then look like this, collected together:

$$\begin{cases} 2s = \varphi(s) + \psi(s) \\ \ \ 1 = -\varphi'(s) + \psi'(s) \qquad s > 0. \\ 2s = \varphi(-s) + \psi(s) \end{cases}$$