COLLECTED WORKS

of A.M. TURING

MATHEMATICAL LOGIC

R.O. Gandy† *&*

C.E.M. Yates

editors

NORTH-HOLLAND

Collected Works of A.M. Turing

MATHEMATICAL LOGIC

Edited by

the late R.O. GANDY
and
C.E.M. YATES
Including prefaces by Solomon Feferman, Andrew Hodges,
Jack Good and Martin Campbell-Kelly

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MATHEMATICAL LOGIC

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Mechanical Intelligence
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Morphogenesis
Edited by P.T. SAUNDERS

Mathematical Logic
Edited by R.O. GANDY and C.E.M. YATES

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(2) As indicated on pages [59], [70], [158], [168] and [186], the following articles have been printed with permission of the Association for Symbolic Logic from: J. Symb. Log. vol. 2 (1937) pp. 153–163 (Computability and λ-definability); ibid. p. 164 (The *p*-function in λ–K conversion), J. Symb. Log. vol. 7 (1942) pp. 28–33 (A formal theorem in Church's theory of types). J. Symb. Log. vol. 7 (1942) pp. 146–156 (The use of dots as brackets in Church's system) and J. Symb. Log. vol. 13 (1948) pp. 80–94 (Practical forms of type theory).

(3) The text of the memoir on Alan Mathison Turing by M.H.A. Newman, which appears on pages [269]–[279], has been reproduced from the Bibliographical Memoirs of Fellows of The Royal Society vol. 1 (November 1955) pp. 253–263 by kind permission of The Royal Society and of Edward and William Newman, the author's executors. The source material for reproducing the photograph accompanying the memoir (on page [268]) is by courtesy of the National Portrait Gallery (London).

Finally, acknowledgement is gratefully made to Ros Moad at the Turing Digital Archive for making available source material for the frontispiece photograph of Alan Turing taken during the thirties. The original is located at King's College Library, Cambridge (copyright holder unknown).

PREFACE

It is not in dispute that A.M. Turing was one of the leading figures in twentieth-century science. The fact would have been known to the general public sooner but for the Official Secrets Act, which prevented discussion of his wartime work. At all events it is now widely known that he was, to the extent that any single person can claim to have been so, the inventor of the "computer". Indeed, with the aid of Andrew Hodges's excellent biography, *A.M. Turing: the Enigma,* even non-mathematicians like myself have some idea of how his idea of a "universal machine" arose – as a sort of byproduct of a paper answering Hilbert's *Entscheidungsproblem.* However, his work in pure mathematics and mathematical logic extended considerably further; and the work of his last years, on morphogenesis in plants, is, so one understands, also of the greatest originality and of permanent importance.

I was a friend of his and found him an extraordinarily attractive companion, and I was bitterly distressed, as all his friends were, by his tragic death – also angry at the judicial system which helped to lead to it. However, this is not the place for me to write about him personally.

I am, though, also his legal executor, and in fulfilment of my duty I have organised the present edition of his works, which is intended to include all his mature scientific writing, including a substantial quantity of unpublished material. The edition will comprise four volumes, i.e.: *Pure Mathematics,* edited by Professor J.L. Britton; *Mathematical Logic,* edited by Professor R.O. Gandy and Professor C.E.M. Yates; *Mechanical Intelligence,* edited by Professor D.C. Ince; and *Morphogenesis,* edited by Professor P.T. Saunders.

My warmest thanks are due to the editors of the volumes, to the modern archivist at King's College, Cambridge, to Dr. Arjen Sevenster and Mr. Jan Kastelein at Elsevier (North-Holland), and to Dr. Einar H. Fredriksson, who did a great deal to make this edition possible.

P.N. FURBANK

ALAN MATHISON TURING – CHRONOLOGY

1912 Born 23 June in London, son of Julius Mathison Turing of the Indian Civil Service and Ethel Sara née Stoney

1926 Enters Sherborne School

1931 Enters King's College, Cambridge as mathematical scholar

1934 Graduates with distinction

1935 Is elected Fellow of King's College for dissertation on the Central Limit Theorem of Probability

1936 Goes to Princeton University where he works with Alonzo Church

1937 (January) His article "On Computable Numbers, with an Application to the Entscheidungsproblem" is published in *Proceedings of the London Mathematical Society*

Wins Procter Fellowship at Princeton

1938 Back in U.K. Attends course at the Government Code and Cypher School (G.C. & C.S.)

1939 Delivers undergraduate lecture-course in Cambridge and attends Wittgenstein's class on Foundations of Mathematics

4 September reports to G.C. & C.S. at Bletchley Park, in Buckinghamshire, where he heads work on German naval "Enigma" encoding machine

1942 Moves out of naval Enigma to become chief research consultant to G.C. & C.S.

In November sails to USA to establish liaison with American code-breakers

1943 January–March at Bell Laboratories in New York, working on speech-encypherment

1944 Seconded to the Special Communications Unit at Hanslope Park in north Buckinghamshire, where he works on his own speech-encypherment project *Delilah*

1945 With end of war is determined to design a prototype "universal machine" or "computer". In June is offered post with National Physical Laboratory at Teddington and begins work on ACE computer

1947 Severs relations with ACE project and returns to Cambridge

1948 Moves to Manchester University to work on prototype computer

1950 Publishes "Computing Machinery and Intelligence" in *Mind*

1951 Is elected FRS. Has become interested in problem of morphogenesis

1952 His article "The Chemical Basis of Morphogenesis" is published in *Philosophical Transactions of the Royal Society*

1954 Dies by his own hand in Wimslow (Cheshire) (7 June)

PREFACE TO THIS VOLUME

Although the unpublished papers were left to Robin Gandy in Turing's will, the matter of inquiring into the possibility of their publication seems to have rested in the first instance with Max Newman[1]. Newman ran into problems with the assessment of some of the unpublished papers, and then on retiring in 1963 he left the whole matter in Robin's hands. It was in fact far too big a task for one person but that was not truly realised until 1988 when Professor Furbank, Turing's executor and overall editor of the Collected Works, stepped in to take a new initiative. There followed an intensive period of activity in which he brought in three additional editors. They did a remarkable job and their volumes were published in 1992.

So why has this, the fourth volume, taken so much longer? As often there is no single reason, but some explanation is due. It was planned at the same time as the other three, becoming the only part of the Collected Works to be edited by Robin. In 1991 I was invited to help him and we made progress, though as expected it was slow. I should at this point make two things clear. First, the scope of the volume envisaged at that point covered only what is now in Parts I and II. Secondly, Robin had in fact written very careful first drafts of prefaces for all of the papers except the 'Ordinal Logics' paper as far back as 1978! He wished to rewrite them and that is what took time. He particularly wanted to rewrite the preface to the famous 1937 paper and to tackle the 'Ordinal Logics' paper. Since Robin had no draft preface for the latter, I did at that time initiate correspondence with Professor Solomon Feferman with a view to possibly using extracts from his excellent paper 'Turing in the Land of O(z)' for this purpose.

Then very sadly, Robin died unexpectedly in November 1995, and so I had to take a fresh look at the volume. I have been especially grateful to Professor Feferman for his prefaces, and for overall advice. Turing's biographer, Dr. Andrew Hodges, has also been particularly helpful. He suggested that this was an opportunity to take some new directions and tidy up some loose ends – hence Part III – and he subsequently provided two prefaces for Part III, including that for the elusive *Enigma* paper written probably in 1940 but not released until 1996.

[1] Professor Maxwell Hermann Alexander Newman, FRS. Newman was involved at certain crucial times in Turing's life. He introduced him to logic in 1935, was also at Bletchley Park and finally brought him to Manchester in 1948. He wrote Turing's Biographical Memoir for the Royal Society, which can be found in this volume.

I am grateful to Professor Jack Good, who worked with Turing at Bletchley Park, and Dr. Martin Campbell-Kelly for the other two prefaces in Part III.

Finally, I must thank Professor Nick Furbank and Dr. Arjen Sevenster for their help and forbearance in what has been a very protracted task.

Mike YATES
July 1999

CONTENTS

Part I
Computability and Ordinal Logics

The historical introduction which opens Part I has been adapted from Feferman's excellent paper [1988]: 'Turing in the Land of O(z)'. The editor wishes to reiterate his gratitude to both Professor Feferman and Oxford University Press, the original publisher of the volume Herken [1988] which includes that paper.

Historical Introduction

by Solomon Feferman

The story of how Turing came to write his papers on computable numbers and ordinal logics is contained in Andrew Hodges' excellent biography, *Alan Turing, The Enigma* [1983]. It is retold in the following in condensed form, drawing extensively on Hodges for the relevant biographical details, as well as Kleene [1981] and Feferman [1986] for the development of logic and recursion theory in this period.

Paper 1. On Computable Numbers with an Application to the Entscheidungsproblem

The story begins with Turing's major achievement, his work on computability, carried out in 1935–6 soon after he became a fellow of King's College Cambridge at the age of 23. In the Spring of 1935 Turing attended a course on the Foundations of Mathematics given by the topologist M.H.A. Newman. Among other things, Newman explained Hilbert's problems concerning consistency, completeness and decidability of various axiomatic systems, as well as Gödel's incompleteness results for sufficiently strong such systems. Turing had already been interested in mathematical logic but had been working primarily on other areas of mathematics, especially group theory. Newman's course served to focus his interests in logic; in particular, Turing became intrigued by the *Entscheidungsproblem* (decision problem) for the first order predicate (or functional) calculus, and this came to dominate his thought from the summer of 1935 on (Hodges [1983], p. 94). In grappling with this problem he was led to conclude that the solution must be negative; but in order to demonstrate that, he would have to give an exact mathematical analysis of the informal concept of *computability by a strictly mechanical process*. This Turing achieved by mid-April 1936, when he delivered a draft of **Paper 1** to Newman. At first Newman was skeptical of Turing's analysis, thinking that nothing so straightforward in its basic conception as the Turing machines could be used to answer this outstanding problem. However, he finally satisfied himself that Turing's notion did indeed provide the most general explanation of finite mechanical process, and he encouraged the paper's publication.

Neither Newman nor Turing were then aware that the question of analyzing the notion of *effective calculability* had occupied the attention of Gödel, Herbrand, and especially Church since the early 1930's. This side of the story is well-told in Kleene [1981], on the origins of recursive function theory.[1] Kleene

[1] Cf. also Gödel [1986], pp. 338–345 and Davis [1982].

was a Ph.D. student of Church from 1931 to 1933 (along with Rosser). Church was promoting a universal system for logic and mathematics in the framework of the lambda (λ)-symbolism for defining functions, and set Kleene the problem of developing the theory of positive integers in his formalism, using an identification of the integers with certain λ-terms. The initial steps were rather difficult (even the predecessor function posed a problem), but once the first hurdles were cleared, Kleene was able to show more and more number-theoretic functions definable by the conversion processes of λ-terms. But Church's original system was shown before long (by Kleene and Rosser in 1934) to be inconsistent, and attention was then narrowed to a demonstrably consistent subsystem, which came to be called the λ-calculus.[2] The consistency of this subsystem was established by Church and Rosser, with some input by Kleene. As it happened, Kleene had already achieved all of his previous definability results in this restricted calculus. It was clear from the Church–Rosser consistency property that every λ-definable function (in the sense given by convertibility in the λ-calculus) is effectively calculable; moreover, Kleene was able to show each example of an effectively calculable function, which came to mind, to be λ-definable. In view of Kleene's success in meeting all such challenges, Church was led to make the proposal, which has come to be known as *Church's Thesis*, that the λ-definable functions comprise all the effectively definable functions. In the words of Kleene: "When Church proposed the thesis, I sat down to disprove it by diagonalising out of the class of the λ-definable functions. But quickly realizing that the diagonalisation cannot be done effectively, I became overnight a supporter of the thesis." (Kleene [1981], p. 59).

Gödel (coming from the University of Vienna) visited the Institute for Advanced Study in Princeton during the year 1933–34, and in the spring of 1934 he gave lectures on his incompleteness results. Notes for these lectures were taken by Kleene and Rosser and, after corrections by Gödel, were circulated at the time. They were subsequently reproduced in Davis' collection [1965] of basic papers on the "undecidable" and computable functions, and more recently in Volume I of Gödel's *Collected Works*. During the course of these lectures, Gödel presented a definition, based on a suggestion of Herbrand, of *general recursiveness* as an analysis of the most general concept of computability, using systems of equations. However, at the time Gödel regarded this identification only as a "heuristic principle".[3] During the same period, Church had conversations with Gödel in which he advanced his own proposal. Gödel regarded this as "thoroughly unsatisfactory" but Church replied that "if [Gödel] would propose any definition of

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[2] There are many variants of the λ-calculus, such as the λ–K calculus, the λ–I calculus etc.; these will not be distinguished here. The classic presentation of this subject is of course in Church [1941]; an up-to-date and comprehensive exposition is provided by Barendregt [1984].

[3] See p. 348 of Gödel [1986].