AN INTRODUCTION TO

HYDRODYNAMICS

AND WATER WAVES

Bernard Le Méhaute

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Preface

Hydrodynamics is the science which deals with the motion of liquid in the macroscopic sense. It is essentially a field which is regarded as applied mathematics because it deals with the mathematical treatments of basic equations for a fluid continuum obtained on a purely Newtonian basis. It is also the foundation of hydraulics, which, as an art, has to compromise with the rigorous mathematical treatments because of nonlinear effects, inherent instability, turbulence, and the complexity of "boundary conditions" encountered in engineering practice. Therefore, this book can be considered as the text for a course in basic hydrodynamics, as well as for a course in the fundamentals of hydraulic and related engineering disciplines.

In the first case, the students learn how to make use of their mathematical knowledge in a field of physics particularly suitable to mathematical treatments. Since they may have some difficulty in representing a physical phenomenon by a mathematical model, a great emphasis has been given to the physical concepts of hydrodynamics. For students with an undergraduate training in engineering, the difficulty may be a lack of appropriate mathematical tools. Their first contact with hydraulics has been on an essentially practical basis. They may be discouraged in attempting the study of such books as *Hydrodynamics*, by Lamb, which remains the bible of hydrodynamicists. Hence, mathematical intricacies have been introduced slowly and progressively. Also, the emphasis on the physical approach has made it possible to avoid mathematical abstractions so that a concrete support may be given to equations.

Finally, the author has tried to make this book self-contained in the sense that a practicing engineer who wants to improve his theoretical background can study hydrodynamics by himself without attending lectures. Too often articles in scientific journals present some discouraging aspects to practicing engineers and the most valuable messages can only reach a few specialists. It is felt that the learning of some basic theories will help hydraulic engineers to keep abreast of and participate in new developments proposed by theorists.

Considering that a good assimilation of the basis is essential before further study, great care has been taken to develop a clear understanding, both mathematically and physically, of the fundamental concepts of theoretical hydraulics. The introduction of mathematical simplifications and assumptions, often based on physical considerations, has also been developed by examples. The mathematical difficulties have been cleared up by introducing them progressively and by developing all the intermediate calculations. Also, all the abstract concepts of theoretical hydraulics have been explained as concretely as possible by use of examples. It will appear that the first chapter is the easiest to understand, and it is assumed that the mathematical background increases as the student progresses toward the end of the book. However, it is taken for granted that the student already has some notion of elementary hydraulics.

Finally, the succession of the various chapters have been chosen in order to build up a structure as logical and as deductive as possible in order to avoid that the various subjects appear as a succession of different mathematical recipes rather than as a unique and logical subject.

Part One deals with the establishment of the fundamental differential equations governing the flow motion in all possible cases. The possible approximations are also indicated. Part Two deals with general methods of integrations and the mathematical treatments of these equations. Integrations of general interest, and integrations in some typical particular cases are presented. Part Three is devoted to water wave theories, as one of the most important topics of hydrodynamics.

It is pointed out that the emphasis of the book is on water waves. Therefore the treatment of motion of compressible fluids has been judged beyond the scope of this book, with a few exceptions. Also, almost all the calculations are presented in a Cartesian (or cylindrical) system of coordinates. Vectorial and tensorial operations have been minimized in order to reduce the necessary mathematical background. However, vectorial and tensorial notations are slowly introduced for sake of recognition in the literature.

It is hoped that this book will entice students gifted in mathematics to apply their capabilities to the study of fluid motion and dynamical oceanography. It is hoped also that it will instill in engineering students the desire for further study in hydrodynamics and mathematics. It is also hoped that the book will be of great help to students in hydraulics, civil and coastal engineering, naval architecture, as well as in physical oceanography, marine geology, and sedimentology, who want to learn or revise one of the theoretical aspect of their future profession.

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PART ONE

Establishing the Basic Equations that Govern Flow Motion

Chapter I

Basic Concepts and Principles

1-1 Basic Concepts of Hydrodynamics

1-1.1 Definition of an Elementary Particle of Fluid

Studies of theoretical fluid mechanics are based on the concept of an elementary mass or particle of fluid. This particle has no well-defined existence. It may be considered as a *corpus alienum*, a foreign body in the mechanics of a continuum. It is an aid toward the understanding of the physical meaning of the differential equations governing the flow motion.

Just as the fundamental concepts of the theoretical mechanics of solid matter are based on the mechanics of a so-called "material point," the basis of theoretical fluid mechanics rests on the mechanics of an elementary mass of fluid. Such an elementary mass of fluid, in common with the material point in the kinematics of a solid body, is assumed to be either infinitely small or small enough that all parts of the element can be considered to have the same velocity of translation $\mathbf{V}$ and the same density ρ. This elementary fluid particle is assumed to be homogeneous, isotropic, and continuous in the macroscopic sense. The molecular pattern and the molecular and Brownian motions within the particle, a subject dealt with in the kinetic theory of fluids, are not taken into account.

1-1.2 Theoretical Approach

The laws of mechanics of a solid body system (a rotating disk, for example) are obtained by the integration of the laws of mechanics for a "material point" with respect to the area or the volume of the system under consideration. Similarly, the laws of fluid mechanics used in engineering practice are obtained by integration—exact or approximate—of the laws governing the behavior of a fluid particle along a line or throughout an area or a volume. Hence, studies in hydrodynamics may be divided into two different parts.

1-1.2.1 The first part consists of establishing the general differential equations which govern the motion of an elementary particle of fluid. The fluid may be assumed either perfect (without friction forces) or real. In the latter case, the flow may be either laminar or turbulent.

1-1.2.2 The second step involves the study of different mathematical methods used to integrate these basic differential equations. Practical general relationships, such as the well-known Bernoulli equation, may thereby be deduced. Solutions, valid for special cases, can also be obtained by direct integration.

1-1.3 Relations between Fluid Particles: Friction Forces

In a solid material, points in a system (on a disk, for example) do not change their relative position (except for elastic deformations which are described by well-defined laws). On the other hand, fluid particles may be deformed and each particle may have a particular motion which differs quite markedly from the motion of other particles. The forces exerted between fluid particles are the pressure forces and the friction forces.

The friction force per unit area in a given direction, called the shear stress τ, is assumed to be either zero ("ideal" or perfect fluid), or proportional to the coefficient of viscosity μ (viscous fluid). The shear stress τ is a scalar. The set of shear stresses at a point constitutes a tensor. The significance of this statement is developed in Chapter 5. For now it is sufficient to know that the shearing stress, at any point of a plane parallel to a unidirectional flow is

$$\tau = \mu \frac{d\mathbf{V}}{dn}$$

where n is the perpendicular direction to the flow moving with velocity $\mathbf{V}$.

Hydrodynamics is primarily concerned with a "Newtonian fluid," that is, its viscous stress tensor depends linearly, isotropically, and covariantly (Chapter 5) on the rate of strain or derivatives of the velocity components. It does not deal with "plastic" fluids where the coefficient μ is replaced by a function of the intensity or duration of the shear.

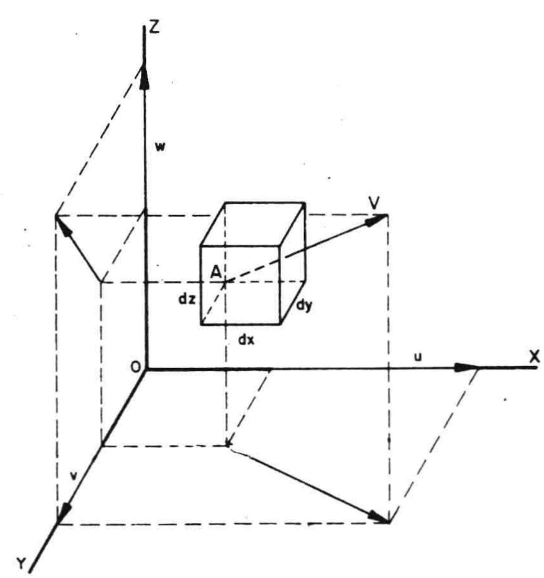
1-2 Streamline, Path, Streakline, and Stream Tube

1-2.1 Notation

Consider the point $A(x,y,z)$ in a Cartesian system of coordinates. The axes OX, OY, OZ are mutually perpendicular (see Fig. 1-1). Consider an infinitely small rectangular element of fluid with the point as a corner. The edges of this element are dx, dy, dz. Its volume is $dx\,dy\,dz$ and its weight is $\varpi\,dx\,dy\,dz$ or $\rho g\,dx\,dy\,dz$. ϖ is the specific weight and g is the acceleration due to gravity.

The pressure at point A is a scalar quantity which is completely specified by its magnitude. The pressure is always exerted perpendicular to the considered surface (see Section 5-3.1). The corresponding force is a vector quantity, which is specified by its magnitude and direction. The magnitude of the pressure p is a function of the space coordinates of A and time t; i.e., $p = f(x,y,z,t)$. Its direction is normal to the area on which the pressure is exerted. The

Figure 1-1 *Notation in Cartesian coordinates.*



gradient of p (**grad** p or ∇p), its derivative with respect to space, is also a vector quantity. The components of **grad** p along the three coordinate axes OX, OY, OZ, are given by the derivative of p with respect to x, y, z, respectively; i.e., $\partial p/\partial x$, $\partial p/\partial y$, $\partial p/\partial z$.

The velocity of fluid particles at A is **V**. The components of **V** along the three Cartesian coordinate axes OX, OY, OZ, are u, v, and w, respectively. If **i**, **j**, **k** are unit vectors along the axes OX, OY, OZ respectively, then: $\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$. Since the system of reference is rectangular, the magnitude of the velocity is given by $V = [u^2 + v^2 + w^2]^{1/2}$. V is a scalar quantity and therefore completely defined by its magnitude, like the pressure p. **V** is a vector quantity and is specified by its direction and magnitude. Since **V** and its components u, v, and w are functions of the space coordinates of A and the time t, they can be written in the form $\mathbf{V}(x,y,z,t)$.

1-2.2 *Definitions*

1-2.2.1 The displacement **dS** of a fluid particle is defined by the vector equation, $\mathbf{dS} = \mathbf{V}\,dt$, which is valid for both magnitude and direction. This equation may be written more specifically in terms of the displacements in each of the three Cartesian coordinate directions as follows:

$$dx = u\,dt$$
$$dy = v\,dt$$
$$dz = w\,dt$$

1-2.2.2 A *streamline* is defined as a line which is tangential at every point to the velocity vector at a given time t_0. A device for visualizing streamlines is to imagine a number of small bright particles distributed at random in the fluid, and then to photograph them with a short exposure (Fig. 1-2). Every particle photographs as a small line segment. Each line which is drawn tangentially to these small segments is a streamline.

At time t_0, the equations $dx = u\,dt$, $dy = v\,dt$, and $dz = w\,dt$ become:

$$\frac{dx}{u(x,y,z,t_0)} = \frac{dy}{v(x,y,z,t_0)} = \frac{dz}{w(x,y,z,t_0)}$$

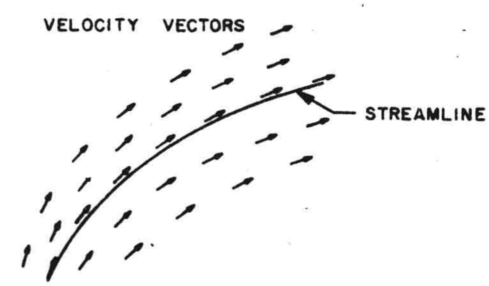


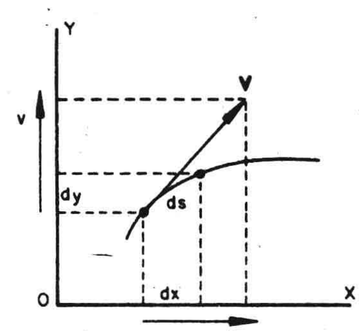
Figure 1-2 *Streamlines observed by short-exposure photography of various particles.*

This is the mathematical definition of a streamline. These equalities express the fact that the velocity is tangential to the displacement of the particle at time t_0. Figure 1-3 illustrates this fact in the case of a two-dimensional motion. In this case $dx/u = dy/v$, which implies $v\,dx - u\,dy = 0$.

Streamlines do not cross, except at point of theoretically infinite velocity (see Figs. 11-6 and 11-7) and at stagnation and separation points of a body where the velocity is zero. Fixed solid boundaries and steady free surfaces are streamlines. Moving boundaries, such as propeller blades, and unsteady free surfaces are not streamlines.

1-2.2.3 The *path* of a specific particle of the fluid is defined by its position as a function of time. It may be

Figure 1-3 *Definition of a streamline in a two-dimensional motion.*



determined by photographing a bright particle with a long exposure. The path line is tangential to the streamline at a given time t_0. However, the time has to be included as a variable for defining a path. Hence, the path lines are defined mathematically as

$$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)} = dt$$

1-2.2.4 A *streakline* is given by an instantaneous shot photographing a number of small bright particles in suspension which were introduced into the fluid at the same point at regular intervals of time (Fig. 1-4).

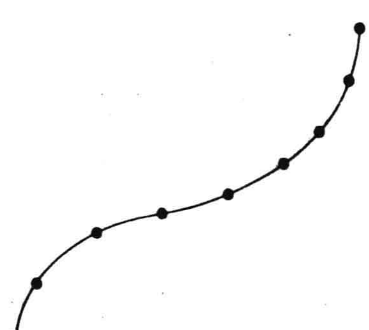
1-2.2.5 An elementary flow channel bounded by an infinite number of streamlines crossing a closed curve is known as a *stream tube* (Fig. 1-5).

1-2.3 Steady and Unsteady Flow

1-2.3.1 For steady flows defined by time-independent quantities, streamlines, streaklines, and particle paths are identical. However, for unsteady or time-dependent flows, these lines are different and a clear understanding of their generation is necessary to properly interpret the results of a given experiment. For example, if dye is injected at a given point of a fluid flow, the dye pattern will be a streakline; if the successive location of a neutrally buoyant small ball are determined, a particle path can be traced; finally, if a large number of short threads are attached to a body, the instantaneous direction of these threads will yield a streamline pattern. All these methods are commonly used in fluid flow experimental studies.

Streamlines, paths, streaklines, and stream tubes are different in unsteady flow, that is, flow changing with respect to time. Turbulent flow is always an unsteady flow; however, it will be seen in that case that the mean motion with respect to time of a turbulent flow may be considered as steady. Then streamlines, paths, and streaklines of the mean motion are the same (see Chapter 7). Figures 1-6 and 1-7 illustrate these definitions in some cases of unsteady motion.

Figure 1-4 *Streakline obtained by instantaneous photography of various particles coming from the same point.*



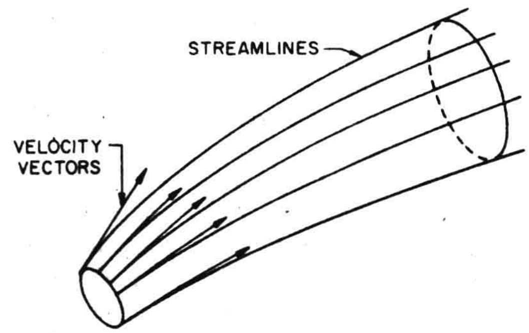


Figure 1-5 *Stream tube.*

1-2.3.2 In some cases of unsteady flow (a body moving at constant velocity in a still fluid, a steady wave profile such as those due to a periodic wave or a solitary wave) it is possible to transform an unsteady motion into a steady motion relative to a coordinate system which moves with the body or the wave velocity. The construction of a steady pattern is then obtained by subtracting the velocity of the body from the velocity of the fluid. This is the *Galilean transformation*. Steady streamlines can then be defined relative to a moving observer who travels with the body or with the wave (see Fig. 1-8).

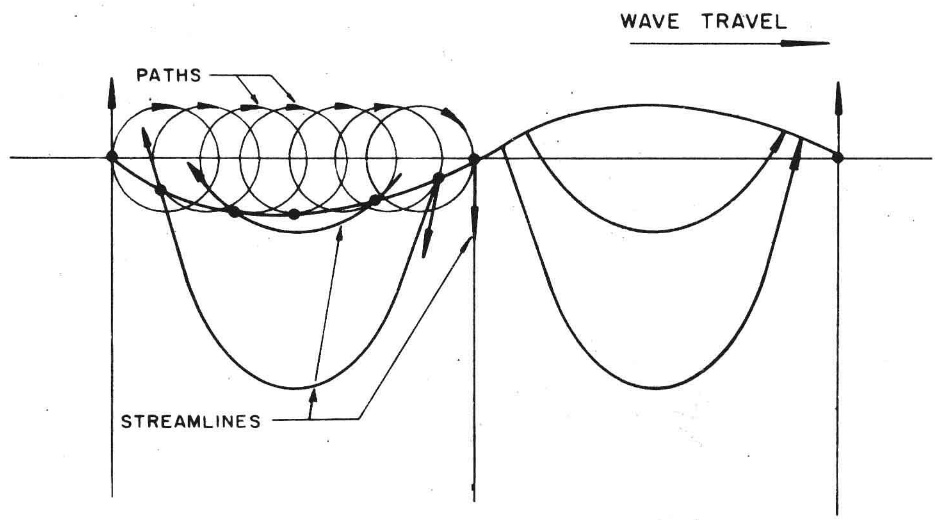


Figure 1-6

Periodic gravity wave in deep water.

1-3 Methods of Study

The motion of a fluid can be studied either by the method of Lagrange or the method of Euler.

1-3.1 Lagrangian Method

The Lagrangian method may be used to answer the question: What occurs to a given particle of fluid as it moves along its own path? This method consists of following the fluid particles during the course of time and giving the paths, velocities, and pressures in terms of the original position of the particles and the time elapsed since the particles occupied their original position. In the case of a compressible fluid, densities and temperatures are also given in terms of the original position and the elapsed time.

If the initial position of a given particle at time t_0 is x_0, y_0, z_0, a Lagrangian system of equations gives the position x, y, z, at the instant t as:

$$x = F_1(x_0, y_0, z_0, t - t_0)$$
$$y = F_2(x_0, y_0, z_0, t - t_0)$$
$$z = F_3(x_0, y_0, z_0, t - t_0)$$

In practice this method is seldom used in hydrodynamics. Lagrangian coordinates are, however, often used in theories relative to periodical gravity waves. The velocity and

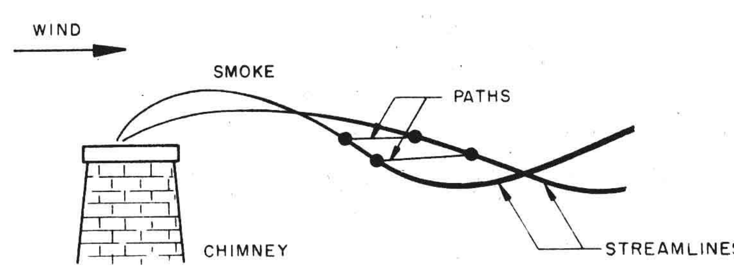
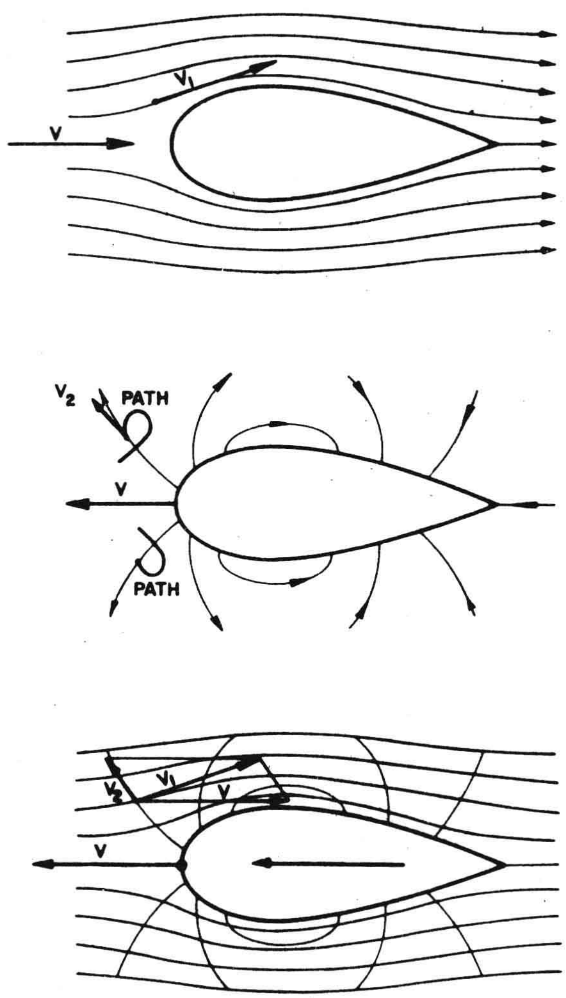


Figure 1-7

Smoke in the wind.

Figure 1-8 (Top) *Streamlines, paths, streaklines for a steady flow around a fixed body.*
(Middle) *Streamlines, paths for an unsteady flow around a body moving at constant velocity in a still fluid.*
(Bottom) *Vectorial relationship between the two kinds of motion: Galilean transformation.*



acceleration components at point (x_0,y_0,z_0) are then obtained by a simple partial differentiation with respect to time, such that

$$u = \left.\frac{\partial x}{\partial t}\right|_{x_0,y_0,z_0}$$

$$v = \left.\frac{\partial y}{\partial t}\right|_{x_0,y_0,z_0}$$

$$w = \left.\frac{\partial z}{\partial t}\right|_{x_0,y_0,z_0}$$

Similarly, the acceleration components are $\partial^2x/\partial t^2$, $\partial^2y/\partial t^2$, $\partial^2z/\partial t^2$.

1-3.2 Eulerian Method

The Eulerian method may be used to answer the question: What occurs at a given point in a space occupied by a fluid in motion? This is the most frequent form of problem encountered in hydrodynamics. This method gives, at a given point $A(x,y,z)$, the velocity $\mathbf{V}(u,v,w)$ and the pressure p (and, in the case of a compressible fluid, density and temperature) as functions of time t. Since

$$\mathbf{V} = \mathbf{F}(x,y,z,t)$$

then

$$u = f_1(x,y,z,t)$$

$$v = f_2(x,y,z,t)$$

$$w = f_3(x,y,z,t)$$

and

$$p = F_1(x,y,z,t)$$

The Eulerian system of equations is found by a total differentiation of u, v, and w with respect to t and by consideration of the pressure components. In the following example the Eulerian system of coordinates is used.

1-3.3 An Example of Flow Pattern

Let us consider an Eulerian system of coordinates where the two-dimensional wave motion is represented by the velocity components:

$$u = f_1(x,z,t) = \frac{dx}{dt} = \frac{H}{2} ke^{mz} \cos (kt - mx)$$

$$w = f_3(x,z,t) = \frac{dz}{dt} = -\frac{H}{2} ke^{mz} \sin (kt - mx)$$

The equations for the streamlines are obtained from the differential equation

$$\frac{dx}{u(x,z,t_0)} = \frac{dz}{w(x,z,t_0)}$$

Thus,

$$\frac{dx}{k(H/2)e^{mz} \cos (kt_0 - mx)} = \frac{dz}{-k(H/2)e^{mz} \sin (kt_0 - mx)}$$

or

$$dz = -\tan (kt_0 - mx)\, dx$$

If t_0 is taken as 0, this equation becomes:

$$dz = -\tan (-mx)\, dx = \tan mx\, dx$$

The integration of this equation gives

$$e^{mz} \cos mx = \text{constant}$$

By varying the value of the constant the streamlines form the general pattern illustrated in Fig. 1-7.

The paths (or particle orbits) are defined by the differential equation:

$$\frac{dx}{u(x,z,t)} = \frac{dz}{w(x,z,t)} = dt$$

where t is a variable. If it can be assumed that x and z differ little from some given values x_0 and z_0, the differential equation, to a first approximation becomes:

$$dx = k \frac{H}{2} e^{mz_0} \cos (kt - mx_0)\, dt$$

Hence,

$$x - x_i = \frac{H}{2} e^{mz_0} \sin (kt - mx_0)$$

$(z - z_i)$ is found by a similar procedure. It is

$$z - z_i = \frac{H}{2} e^{mz_0} \cos (kt - mx_0).$$

In order to eliminate t, square the equations for $(x - x_i)$ and $(z - z_i)$ and add the results. This gives:

$$(x - x_i)^2 + (z - z_i)^2 = \left[\frac{H}{2} e^{mz_0}\right]^2$$

This is the equation of a circle of radius $(H/2)e^{mz_0}$. It is seen that the paths are circular and the radius tends to zero as $z_0 \to -\infty$. It will be seen in the linear wave theory that, at a first approximation, one has $x_i \cong x_0$, $z_i \cong z_0$ (Section 16-1), and x_0, z_0 can be considered the location of the fluid particle at rest.

1-4 Basic Equations

1-4.1 The Unknowns in Fluid Mechanics Problems

In general, the density of a liquid is assumed constant so that equations are needed only for velocity and pressure. Hence, in the Eulerian system of coordinates, the motions are completely known at a given point x, y, z if one is able to express $\mathbf{V}$ and p as functions of space and time: $\mathbf{V} = \mathbf{F}(x,y,z,t)$ and $p = F_1(x,y,z,t)$. Therefore, to solve problems in hydrodynamics two equations are necessary, one of them being vectorial. If **V** is expressed by its components u, v, and w, four scalar or ordinary equations are necessary.

In free surface flow problems, the free surface elevation $\eta(x,y,z,t)$ around the still water level, or the water depth $h(x,y,z,t)$, is unknown and a kinematic condition is also required. However, in that case the pressure p is known and in general is equal to the atmospheric pressure.

For gases, two more unknowns need to be considered, namely, the density ρ and the absolute temperature T. Hence, to solve problems in the most general cases of fluid mechanics, four equations are necessary. If **V** is expressed by u, v, and w, then six ordinary equations are needed.

In hydrodynamics, basic equations are given by the physical principles of continuity and conservation of momentum. The equation of state and the principle of the conservation of energy must be added in the case of compressible fluid.

The reduction of a problem to such a small number of variables (2 in hydrodynamics and 4 in gas dynamics), does not occur for trivial reasons, but as a result of several important arguments and assumptions. A number of phenomenological functions are assumed to be known. For example, it is assumed that the fluid is Newtonian and either perfect or viscous, which defines the stress tensor. The fluid obeys Fourier's law of conduction. Also, a number of coefficients, such as heat conductivity, specific heat, and viscosity, are supposed to be known functions of the other unknown variables, such as density and/or temperature.

1-4.2 Principle of Continuity

The continuity principle expresses the conservation of matter, i.e., fluid matter in a given space cannot be created or destroyed. In the case of an incompressible homogeneous fluid, the principle of continuity is expressed by the conservation of volume, except in the special case of cavitation where partial voids appear.

The continuity principle gives a relationship between the velocity **V**, the density ρ, and the space coordinates and time. If ρ is constant (in the case of an incompressible fluid), it gives a relationship between the components of **V** and the space coordinates, which are x, y, z. The equation of continuity then becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

as it is demonstrated in Section 3-2.

It will be seen that **V** may be found in some cases of flow under pressure, independent of the absolute value for p, from the principle of continuity alone, but p will always be a function of **V** except at the free surface.

1-4.3 The Momentum Principle

The momentum principle expresses the relationship between the applied forces **F** on a unit volume of matter of density ρ and the inertia forces $d(\rho \mathbf{V})/dt$ of this unit volume of matter in motion. The inertia forces are due to the natural tendency of bodies to resist any change in their motion. It is Newton's first law that "every body continues in its state of rest or uniform motion via a straight line unless it is compelled by an external force to change that state." The well-known Newtonian relationship is derived from his second law: "The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts." $\mathbf{F} = d(m\mathbf{V})/dt$.

In fluid mechanics this equation takes particular forms which take into account the fact that the fluid particle may be deformed. These equations will be studied in detail. For an incompressible fluid, the integration of the momentum equation with respect to distance gives an equality of work and energy, expressing a form of the conservation of energy principle.

If **V** is expressed by u, v, w, then Newton's second law has to be expressed along the three coordinate axes. This gives the three equations

$$F_x = \rho \frac{du}{dt} \qquad F_y = \rho \frac{dv}{dt} \qquad F_z = \rho \frac{dw}{dt}$$

where ρ is assumed constant and F_x, F_y, F_z are the components of **F** along the three coordinate axes, respectively.