Arsen Melkumian

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Mathematical Economics

Mathematics and mathematical economics are subjects foreign to many students. The concepts covered in the average course are lost on many without the opportunity for interactive practice. Matlab and Mathematica® are software packages that enable students to see mathematics come alive on their screens. *Mathematical Economics* serves as a guide to the essential mathematics of economics and as an introduction to the powerful tools represented by those software packages.

This textbook, designed for a single-semester course, begins with basic set theory, and moves briskly through fundamental, exponential and logarithmic functions. Limits and derivatives finish the preparation for economic applications, which are introduced in chapters on univariate functions, matrix algebra, and the constrained and unconstrained optimization of univariate and multivariate functions. The text finishes with chapters on integrals, the mathematics of finance, complex numbers, and differential and differences equations.

Rich in targeted examples and explanations, *Mathematical Economics* offers the utility of a handbook and the thorough treatment of a text. While the typical economics text is written for two-semester applications, this text is focused on the essentials. Instructors and students are given the concepts in conjunction with specific examples and their solutions. Typical mathematical economics texts do not introduce software packages; *Mathematical Economics* introduces two, Matlab and Mathematica. It is the pairing of specific textbook treatments with interactive software that sets this text apart and allows it to be used in a variety of course formats and as a reference.

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Preface

This book is intended to cover most of the mathematics that would be typically used in undergraduate and lower-level graduate economics courses. The background recommended for readers would be college-level algebra and the knowledge normally acquired in courses on the principles of economics. This book contains numerous examples, visually stimulating graphs and representations. I want especially to point out the need to integrate mathematical knowledge with functional software in today's ever-evolving economic landscape. At the end of many of the chapters I include examples utilizing Mathematica or Matlab. I chose Mathematica over other software packages because of its ability to combine highly sophisticated functions with wonderful usability.

One of the aims of this book is to give aspiring economists the basic mathematical tools they need to excel at all levels of their discipline. I expect the students to learn economic theory in other courses using the mathematical and programming skills acquired in the process of studying this text. The material starts out with basic set theory and runs the gamut of mathematical economics, including univariate and multivariate optimization and some mathematics of finance. Each section comes with its own carefully chosen examples, which teach the material without overwhelming the reader with useless repetition. The logical progression of topics covered should help the reader to master the material presented in this book.

The answers to odd-numbered problems are included at the end of the book. The instructor's manual, which includes answers to the even-numbered problems, is available for those professors, lecturers, and teachers who choose to adopt this textbook.

Acknowledgements

I would like to thank my colleagues from the Departments of Economics and Mathematics at Western Illinois University for performing various editorial tasks. I am especially grateful to my students, who have provided valuable comments and spotted errors. I would also like to express my deep appreciation to my family and friends for making this text possible.

Finally, I would like to thank the editors at Routledge for their excellent guidance.

Contents

1 Introduction

The purpose of this chapter is to familiarize the reader with basic concepts in set theory and introduce our notation. The first section introduces sets of natural, rational, irrational and real numbers and provides several examples of relations. The following section considers the polynomial functions frequently used by economists, linear, quadratic and cubic, and demonstrates how to minimize a quadratic profit function (or any quadratic function for that matter). The concepts introduced in this chapter will be used frequently, so a thorough understanding of the chapter's material is essential to the remainder of this book.

1.1 Basic set theory

The numbers that we use for counting, namely $1, 2, 3, \ldots$, are called *natural numbers*. Number 1 is the lowest natural number and there are infinitely many natural numbers. A collection of natural numbers, such as 1, 7, 25 and 46, is referred to as a set.

In general, a *set* is a collection of objects. These objects may be natural numbers, coins, movies, people and so on. The objects in a set are referred to as the elements of the set. In some cases we can define a set by enumeration of the elements. For instance, natural numbers 2, 7, 11 and 23 can form a set

$$S = \{2, 7, 11, 23\}$$

In this case, we have defined our set S by enumeration of the elements. The set S is finite, as it contains a finite number of elements. When a set is infinite or contains a lot of elements, we are forced to define it by description. For example, the set

$$W = \{x \mid x \text{ is a natural number greater than } 5\}$$

reads as follows: "W is the set of all natural numbers x, such that x is greater than five." The vertical bar in the description of W simply means "such that." Set W is infinite, as there are infinitely many natural numbers greater than five.

If the elements of a set can be counted using natural numbers $1, 2, 3, 4$ and so on, then the set is considered to be countable. In particular, set S as well as any other finite set is countable. Infinite sets may be countable or uncountable. For example, set W is countable since its elements can be counted using natural numbers: the first element of W is 6, the second element of W is 7 and so on. The set of natural numbers, referred to as $\mathbb{N}$ and given by

$$\mathbb{N} = \{x \mid x \text{ is a natural number}\}$$

is also countable.

Now, let us define the set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Clearly, $\mathbb{Z}$ is an infinite set that is countable. The set of integers $\mathbb{Z}$ is also known as the set of whole numbers. We shall write $\mathbb{Z}^+$ or $\mathbb{Z}^-$ to refer to the set of positive or negative integers, respectively.

EXAMPLE 1.1 Show that the set of all integers $\mathbb{Z}$ is countable.

Solution: The set of integers can be counted as follows: the first element of $\mathbb{Z}$ is 0, the second element of $\mathbb{Z}$ is 1, the third element of $\mathbb{Z}$ is -1, the fourth element of $\mathbb{Z}$ is 2, the fifth element of $\mathbb{Z}$ is -2 and so on. The set is countable.

Now, consider another set:

$$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers with no common divisors} \right\}$$

This is called the set of rational numbers. Rational numbers are simply fractions in lowest terms. It could be shown that the set of rational numbers is countable. The set of rational numbers $\mathbb{Q}$ contains both the set of integers $\mathbb{Z}$ and the set of natural numbers $\mathbb{N}$. Alternatively, we can say that the set of integers $\mathbb{Z}$ is a subset of $\mathbb{Q}$ and the set of natural numbers is a subset of $\mathbb{Q}$ and write

$$\mathbb{Z} \subset \mathbb{Q}$$

and

$$\mathbb{N} \subset \mathbb{Q}$$

to indicate that fact. In general, if every element of set A is also an element of set B, we write

$$A \subset B$$

to indicate that A is a subset of B. Further, set A is said to be a proper subset of set B if

$$A \subset B \text{ and } A \neq B$$

Note that $\mathbb{N}$ and $\mathbb{Z}$ are proper subsets of $\mathbb{Q}$. Also, the set

$$S = \{2, 7, 11, 23\}$$

is a proper subset of $\mathbb{Q}$. In fact, since every element of A is also an element of A, we have $A \subset A$.

In set theory, there is a unique set $\varnothing$ that contains no elements. This set is called the empty set and we can easily prove that the empty set is a subset of any set.

EXAMPLE 1.2 Show that the empty set ∅ is a subset of an arbitrary set A.

Solution: The empty set contains no elements. Therefore we can say that if a is an element of the empty set, then a is also an element of set A and write

$$\text{if } a \in \varnothing \quad \text{then } a \in A$$

The sign ∈ reads as "belongs to" and

$$a \in A$$

reads as "a belongs to A" or "a is an element of A" or simply "a is in A." We write

$$a \notin A$$

to indicate that a does not belong to A. Integers and rational numbers can be represented on a number line like the one shown in Figure 1.1.

Since there are lots of rational numbers, it would appear that rational numbers could cover the number line. In other words, it would appear that every point on the real line could be assigned to a rational number. The ancient Greeks found that not every number on the number line is rational. In fact, it could be shown that $\sqrt{3}$ is not a rational number. Furthermore, there are infinitely many numbers on the number line that are not rational.

Numbers on the number line that are not rational are called *irrational*. There are infinitely many irrational numbers and the set of irrational numbers is not countable. Intuitively speaking, the set of irrational numbers contains a lot more elements than the set of rational numbers. The *set of real numbers* $\mathbb{R}$ consists of rational and irrational numbers and is uncountable: there is no way to associate elements of $\mathbb{R}$ with the elements of $\mathbb{N}$. In other words, the set of real numbers $\mathbb{R}$ is the union of the set of rational numbers $\mathbb{Q}$ and the set of irrational numbers. There is also a set of complex numbers, which contains all real numbers. Complex numbers are introduced as pairs of real numbers. Complex numbers are defined in Chapter 11 and most of this text deals with real numbers only. Unless otherwise specified, from now on by a "number" we shall understand a real number.

By definition, the *union of two sets* A and B, denoted $A \cup B$, is the set of elements contained in either A or B. Thus:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

For example, if $A = \{2, 3, 7\}$ and $B = \{3, 11, 12\}$ then $A \cup B = \{2, 3, 7, 11, 12\}$.

The *intersection of two sets* A and B, denoted by $A \cap B$, is the set of elements contained in both A and B. Thus:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

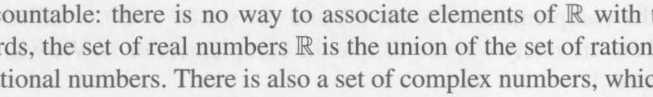

Figure 1.1

For example, if $A = \{2, 3, 7\}$ and $B = \{3, 11, 12\}$, then $A \cap B = \{3\}$. Sets A and B have one element in common. If two sets have no elements in common ($A \cap B = \varnothing$), they are called disjoint.

The *difference of two sets A* and *B*, denoted by $A - B$, is the set of elements that belong to A but not to B. Thus:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 7\}$, then $A - B$ is equal to

$$A - B = \{1, 2, 3\}$$

When considering a collection of sets, say $A = \{2, 3, 7\}$ and $B = \{3, 11, 12\}$, it is useful to think of each set in the collection as a subset of some set U called the universal set. The choice of U depends only on the particular problem at hand. Let us say

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

then with A and B as before, we have

$$U - A = \{1, 4, 5, 6, 8, 9, 10, 11, 12\}$$

and

$$U - B = \{1, 2, 4, 5, 6, 7, 8, 9, 10\}$$

The set $(U - A)$ is called the complement of A (in U) and the set $(U - B)$ is called the complement of B (in U). There are many ways to denote the complement of A. We will use $\bar{A}$ to denote the complement of A.

The power set of a set A is defined as the set of all subsets of A and is denoted as $P(A)$.

EXAMPLE 1.3 Find the power set of $A = \{1, 2\}$.

Solution: The power set of A is $P(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

1.1.1 Venn diagrams

Unions, intersections and complements of sets can sometimes be visualized using so-called Venn diagrams. In Figure 1.2, the points on and inside the circle form set A and the points on and inside the oval form set B. The intersection of A and B consists of the shaded area. The points in the shaded area belong to both A and B.

1.1.2 Cartesian product

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all possible ordered pairs whose first component is a member of A and whose second component is a member of B. Thus:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

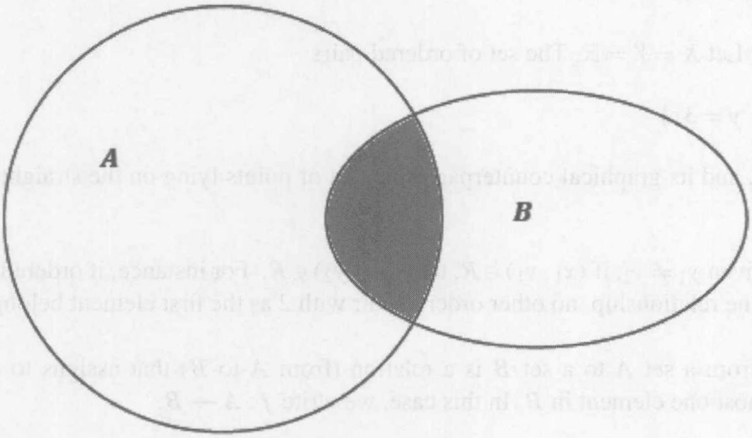

Figure 1.2

For example, if $A = \{2, 3\}$ and $B = \{4, 5, 6\}$, then

$$A \times B = \{(2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$$

EXAMPLE 1.4 Let $A = B = \mathbb{R}$. Then $A \times B = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. The set $\mathbb{R}^2$ is often referred to as the real plane.

1.1.3 Relations

Let A and B be sets. A relation, R, from A to B is a subset of $(A \times B)$. Now, suppose $x \in A$ and $y \in B$. The notations xRy (x is in relation R to y) and $(x, y) \in R$ are equivalent. If $A = B$, then a relation R from A to B is known as a relation on A.

EXAMPLE 1.5 Let $A = B = \mathbb{R}$. Define a relation R from A to B as follows: xRy if $x = y$. Then $(x, y) \in R$ implies that $x = y$.

1.1.4 Elementary types of relations on X

Let us consider a set X and a relation R on set X. A relation R on set X is said to be

(a) reflexive if xRx,
(b) symmetric if $xRy \Rightarrow yRx$,
(c) transitive if (xRy and yRz) $\Rightarrow xRz$,
(d) complete if either xRy or yRx,

for all $x, y, z \in X$.

A relation on X may have any combination of these four properties, including none.

EXAMPLE 1.6 Consider the strict "less than" relation R_1 on a set of natural numbers $\mathbb{N}$. R_1 is not reflexive, since $n < n$ is not true. The relation R_1 is not symmetric, since $n < m$

does not imply $m < n$ for $n, m \in \mathbb{N}$. However, R_1 is transitive since $n < m$ and $m < w$ implies $n < w$ for all $n, m, w \in \mathbb{N}$.

EXAMPLE 1.7 Let $X = Y = \mathbb{R}$. The set of ordered pairs

$$\{(x, y) \in \mathbb{R}^2 \mid y = 3x\}$$

is a relation on $\mathbb{R}$, and its graphical counterpart is the set of points lying on the straight line $y = 3x$.

Observe that, given $y_1 \neq y_2$, if $(x_1, y_1) \in R$, then $(x_1, y_2) \notin R$. For instance, if ordered pair $(2, 6)$ belongs to the relationship, no other ordered pair with 2 as the first element belongs to the relationship.

A *function* f from a set A to a set B is a relation (from A to B) that assigns to each element in A at most one element in B. In this case, we write $f \colon A \to B$.

EXAMPLE 1.8 Consider a function f from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$ given by $f(x) = \sqrt{x}$. Here $f(4) = 2$ but $f(4.5)$ is undefined, since 4.5 is not a natural number.

Now, if $f \colon A \to B$, the subset of A on which f is defined is called the *domain* of f; and the subset of B in which f takes its values is called the *range* of f.

EXAMPLE 1.9 Consider a function $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then, the domain of f is all of $\mathbb{R}$, written

$$\text{Domain}(f) = \mathbb{R}$$

and the range of f is the set of all non-negative real numbers, written

$$\text{Range}(f) = \{y \in \mathbb{R} \mid y \geq 0\}$$

1.1.5 Necessary and sufficient conditions

To better understand when certain mathematical methods can be used in applied situations, it is important to remember that every method has its limitations. The conditions that must be satisfied in order for us to use a certain formula are usually stated in a mathematical statement known as a theorem. To read theorems properly, we will remind the reader about the distinction between necessary and sufficient conditions.

Consider the statement: "It is necessary to graduate from a high school to enter a Ph.D. program." The statement means that, unless you have graduated from a high school, you cannot enter a Ph.D. program. It is a necessary condition. However, it is by no means sufficient to graduate from high school to be accepted into a Ph.D. program. The statement does *not* say that, if you are a high-school graduate, then you can enter a Ph.D. program.

Now consider the statement: "If n is a natural number, then n is a real number." The statement is equivalent to saying the following: in order for a number to be real, it is sufficient for it to be natural. By no means does it require the number to be natural: as we know there are real numbers that are not natural. This is an example of a sufficient condition.

If from a logical statement p we can deduce a logical statement q, then we will express this fact as $p \Rightarrow q$. We say "if p then q" or "q follows from p." If in addition p follows from q, then the statements p and q are logically equivalent and we write $p \Leftrightarrow q$. The logical equivalence statement $\Leftrightarrow$ means that p happens if and only if q does. Quite often an abbreviation "if and only if" is used instead of saying "if and only if." Sometimes the symbol $\Leftrightarrow$ is used.

Many times in the text we make statements valid for any object in a certain class, for instance every real number. The symbol $\forall$ is used as a short-cut for "for any." For instance,

$$\forall\, x > 0$$

reads "for any $x > 0$."

PROBLEMS FOR SECTION 1.1

1. Write the following in set notation.

   (a) the set of all real numbers
   (b) the set of all natural numbers
   (c) the set of all non-negative integers
   (d) the set of all real numbers greater than 76
   (e) the set of all natural numbers less than 22
   (f) the set that contains no elements

2. Let $A = \{1, 2, 3\}$, $B = \{3, 4, 7\}$, $C = \{3, 4\}$ and $D = \{7\}$.

   (a) Determine which of the following statements are true.

   (i) $5 \in A$
   (ii) $7 \in B$
   (iii) $D \subset B$
   (iv) $B \subset D$
   (v) $A \subset B$

   (b) Find the following sets.

   (i) $A \cap B$
   (ii) $A \cup B$
   (iii) $A - B$
   (iv) $B - A$
   (v) $(A \cap B) - A$
   (vi) $(B \cup A) - B$
   (vii) $(A \cup B) - (C \cup D)$
   (viii) $A \cap B \cap C \cap D$
   (ix) $A \cap C \cap D \cap \varnothing$
   (x) $((A \cup B) - C) - D$

3. Describe the power set of each of the following.

   (a) $B_1 = \{a\}$
   (b) $B_2 = \{a, b\}$
   (c) $B_3 = \{1, 2, 3\}$

(d) $B_4 = \{5, 7, 9, 11\}$
(e) $B_5 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

4. Which of the following statements are valid?

(a) $A \cup A = A$
(b) $A \cup \varnothing = A$
(c) $A \cup U = A$
(d) $A \cap A = A$
(e) $A \cap \varnothing = U$
(f) $A \cap U = U$
(g) $\bar{\bar{A}} = A$

5. Prove that $A \subset B$ and $B \subset A$ imply that $A = B$.
6. A survey revealed that 80 students liked economics and 70 students liked finance. Further, 20 students liked both economics and finance, and 50 students liked neither. How many students responded to the survey?

1.2 Functions from $\mathbb{R}$ to $\mathbb{R}$

1.2.1 Polynomial functions

Most likely the simplest function that you will encounter is the constant function $f(x) = a$, where a is a constant. For example, $f(x) = 17.5$ (or alternatively $y = 17.5$) is a constant function. The value of the function y is always equal to 17.5 regardless of the value of x (see Figure 1.3). In this case, we have Domain$(f) = \mathbb{R}$ and Range$(f) = \{17.5\}$.

A constant function is a special case of a polynomial function. The general form of a polynomial function is

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

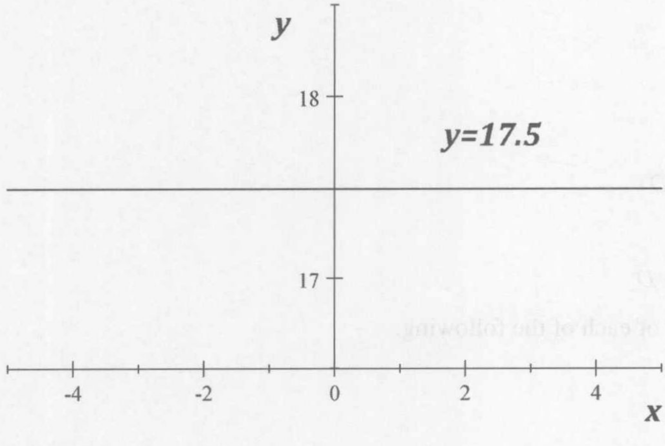

Figure 1.3

Fixing $n=0$, we have $y=a_0$. Letting $n=1, 2, 3, 4$, in turn, we have

$y=a_0+a_1x$	linear
$y=a_0+a_1x+a_2x^2$	quadratic
$y=a_0+a_1x+a_2x^2+a_3x^3$	cubic
$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$	quartic

Linear functions yield graphs that are straight lines, hence the name linear. Recall that the equation of a linear function is given by $y=a_0+a_1x$, where a_0 is the intercept and a_1 is the slope. Examples of linear functions include the following:

$$y_1=2x+5$$
$$y_2=-3x+7$$
$$y_3=7x$$

The functions above are represented graphically in Figure 1.4.

1.2.2 Zeros of quadratic functions

DEFINITION 1.1 The zeros of a function are the x-values where the graph of the function intersects the x-axis. Algebraically, zeros of function $f(x)$ are numbers x_i such that $f(x_i)=0$.

A polynomial function of degree n has n zeros, provided multiple and complex zeros are counted. The zeros of polynomial functions are also known as roots.

A quadratic function is given by the following equation:

$$y=a_0+a_1x+a_2x^2, \quad \text{where } a_2 \neq 0$$

Figure 1.5 depicts the simplest ($a_0=0,\ a_1=0,\ a_2=1$) quadratic function $y=x^2$.

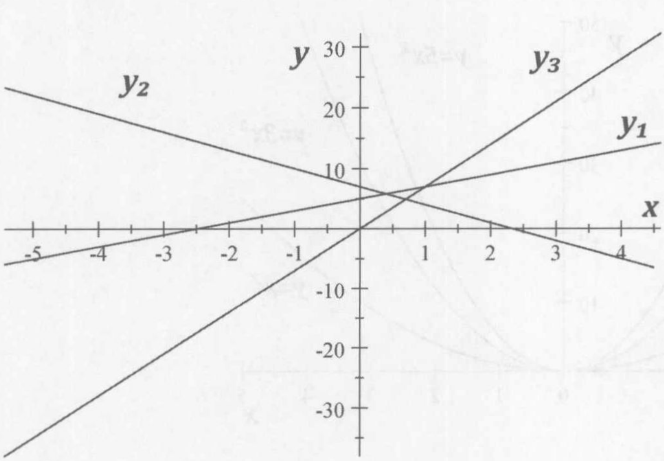

Figure 1.4