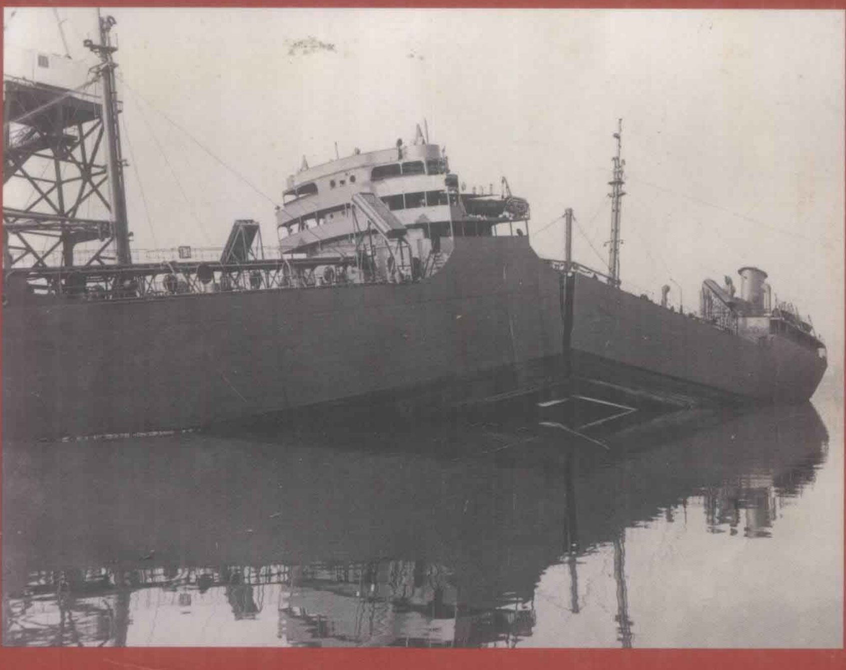
SOLID MECHANICS



WILLIAM F. HOSFORD

Solid Mechanics

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SOLID MECHANICS

This is a textbook for courses in departments of Mechanical, Civil and Aeronautical Engineering commonly called strength of materials or mechanics of materials. The intent of this book is to provide a background in the mechanics of solids for students of mechanical engineering while limiting the information on why materials behave as they do. It is assumed that the students have already had courses covering materials science and basic statics. Much of the material is drawn from another book by the author, *Mechanical Behavior of Materials*. To make the text suitable for Mechanical Engineers, the chapters on slip, dislocations, twinning, residual stresses, and hardening mechanisms have been eliminated and the treatments in other chapters about ductility, viscoelasticity, creep, ceramics, and polymers have been simplified.

William Hosford is a Professor Emeritus of Materials Science at the University of Michigan. He is the author of numerous research and publications books, including *Materials for Engineers; Metal Forming* third edition (with Robert M. Caddell); *Materials Science: An Intermediate Text; Reporting Results* (with David C. Van Aken); *Mechanics of Crystals and Textured Polycrystals; Mechanical Metallurgy*; and *Wilderness Canoe Tripping*.

Preface

The intent of this book is to provide a background in the mechanics of solids for students of mechanical engineering without confusing them with too much detail on why materials behave as they do. The topics of this book are similar to those in *Deformation and Fracture of Solids* by R. M. Caddell. Much of the material is drawn from another book by the author, *Mechanical Behavior of Materials*. To make the text suitable for Mechanical Engineers, the chapters on slip, dislocations, twinning, residual stresses, and hardening mechanisms have been eliminated and the treatments in other chapters about ductility, viscoelasticity, creep, ceramics, and polymers have been simplified. If there is insufficient time or interest, the last two chapters, "Mechanical Working" and "Anisotropy," may be omitted. It is assumed that the students have already had courses covering materials science and basic statics.

I want to thank Professor Robert Caddell for the inspiration to write texts. Discussions with Professor Jwo Pan about what to include were helpful.

Conversions

To convert from	To	Multiply by
inch, in.	meter, m	0.0254
pound force, lb_f	newton, N	0.3048
pounds/inch^2	pascal, Pa	6.895×10^3
kilopound/inch^2	megapascal, MPa	6.895×10^3
kilograms/mm^2	pascals	9.807×10^6
horsepower	watts, W	7.457×10^2
horsepower	ft-lb/min	33×10^3
foot-pound	joule, J	1.356
calorie	joule, J	4.187

SI Prefixes

tera	T	10^{12}	pico	p	10^{-12}
giga	G	10^9	nano	n	10^{-9}
mega	M	10^6	micro	μ	10^{-6}
kilo	k	10^3	milli	m	10^{-3}

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1 Stress and Strain

Introduction

This book is concerned with the mechanical behavior of materials. The term *mechanical behavior* refers to the response of materials to forces. Under load, materials may either deform or break. The factors that govern a material's resistance to deforming are very different than those governing its resistance to fracture. The word *strength* may refer either to the stress required to deform a material or to the stress required to cause fracture; therefore, care must be used with the term *strength*.

When a material deforms under a small stress, the deformation may be *elastic*. In this case when the stress is removed, the material will revert to its original shape. Most of the elastic deformation will recover immediately. However, there may be some time-dependent shape recovery. This time-dependent elastic behavior is called *anelasticity* or *viscoelasticity*.

A larger stress may cause *plastic* deformation. After a material undergoes plastic deformation, it will not revert to its original shape when the stress is removed. Usually, a high resistance to deformation is desirable so that a part will maintain its shape in service when stressed. On the other hand, it is desirable to have materials deform easily when forming them into useful parts by rolling, extrusion, and so on. Plastic deformation usually occurs as soon as the stress is applied. At high temperatures, however, time-dependent plastic deformation called *creep* may occur.

Fracture is the breaking of a material into two or more pieces. If fracture occurs before much plastic deformation occurs, we say the material is *brittle*. In contrast, if there has been extensive plastic deformation preceding fracture, the material is considered *ductile*. Fracture usually occurs as soon as a critical fracture stress has been reached; however, repeated applications of a somewhat lower stress may cause fracture. This is called *fatigue*.

The amount of deformation that a material undergoes is described by *strain*. The forces acting on a body are described by *stress*. Although the reader

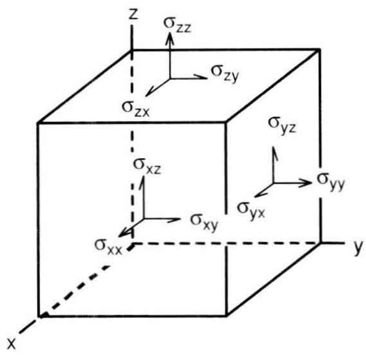


Figure 1.1. The nine components of stress acting on an infinitesimal element. The normal stress components are σ_{xx}, σ_{yy}, and σ_{zz}. The shear stress components are $\sigma_{yz}, \sigma_{zx}, \sigma_{xy}, \sigma_{zy}, \sigma_{xz}$, and σ_{yx}.

should already be familiar with these terms, they will be reviewed in this chapter.

Stress

Stress, σ, is defined as the intensity of force at a point,

$$\sigma = \partial F/\partial A \text{ as } \partial A \to 0. \tag{1.1a}$$

If the state of stress is the same everywhere in a body,

$$\sigma = F/A. \tag{1.1b}$$

A *normal stress* (compressive or tensile) is one in which the force is normal to the area on which it acts. With a *shear stress*, the force is parallel to the area on which it acts.

Two subscripts are required to define a stress. The first subscript denotes the normal to the plane on which the force acts, and the second subscript identifies the direction of the force.* For example, a tensile stress in the x-direction is denoted by σ_{xx}, indicating that the force is in the x-direction and it acts on a plane normal to x. For a shear stress, σ_{xy}, a force in the y-direction acts on a plane normal to x.

Because stresses involve both forces and areas, they are tensor rather than vector quantities. Nine components of stress are needed to describe fully a state of stress at a point, as shown in Figure 1.1. The stress component $\sigma_{yy} = F_y/A_y$ describes the tensile stress in the y-direction. The stress component $\sigma_{zy} = F_y/A_z$ is the shear stress caused by a shear force in the y direction acting on a plane normal to z.

Repeated subscripts denote normal stresses (e.g. $\sigma_{xx}, \sigma_{yy}, \ldots$), whereas mixed subscripts denote shear stresses (e.g. $\sigma_{xy}, \sigma_{zx}, \ldots$) . In *tensor* notation,

* Use of the opposite convention should cause no confusion as $\sigma_{ij} = \sigma_{ji}$.

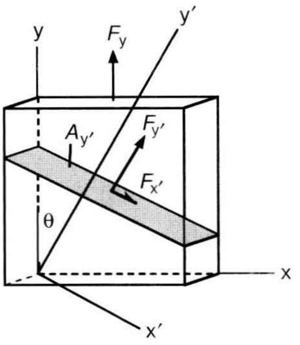


Figure 1.2. Stresses acting on an area, A', under a normal force, F_y. The normal stress is $\sigma_{y'y'} = F_{y'}/A_{y'} = F_y\cos\theta/(A_y/\cos\theta) = \sigma_{yy}\cos^2\theta$. The shear stress is $\tau_{y'x'} = F_x/A_{y'} = F_y\sin\theta/(A_{yx}/\cos\theta) = \sigma_{yy}\cos\theta\sin\theta$.

the state of stress is expressed as

$$\sigma_{ij} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{vmatrix}, \tag{1.2}$$

where i and j are iterated over x, y, and z. Except where tensor notation is required, it is often simpler to use a single subscript for a normal stress and to denote a shear stress by τ,

$$\sigma_x = \sigma_{xx}, \quad \text{and} \quad \tau_{xy} = \sigma_{xy}. \tag{1.3}$$

A stress component, expressed along one set of axes, may be expressed along another set of axes. Consider the case in Figure 1.2. The body is subjected to a stress $\sigma_{yy} = F_y/A_y$. It is possible to calculate the stress acting on a plane whose normal, y', is at an angle θ to y. The normal force acting on the plane is $F_{y'} = F_y\cos\theta$ and the area normal to y' is $A_y/\cos\theta$, so

$$\sigma_{y'} = \sigma_{y'y'} = F_{y'}/A_{y'} = (F_y\cos\theta)/(A_y/\cos\theta) = \sigma_y\cos^2\theta. \tag{1.4a}$$

Similarly, the shear stress on this plane acting in the x' direction, $\tau_{y'x'}(= \sigma_{y'x'})$, is given by

$$\tau_{y'x'} = \sigma_{y'x'} = F_{x'}/A_{y'} = (F_y\sin\theta)/(A_y/\cos\theta) = \sigma_y\cos\theta\sin\theta. \tag{1.4b}$$

Note that the transformation equations involve the product of two cosine and/or sine terms.

Sign Convention

When we write $\sigma_{ij} = F_i/A_j$, the term σ_{ij} is positive if i and j are either both positive or both negative. On the other hand, the stress component is negative for a combination of i and j in which one is positive and the other is negative. For example, in Figure 1.3 the term σ_{xx} is positive on both sides of the element because both the force and normal to the area are negative on the left and positive on the right. The stress τ_{yx} is negative because on the top surface y is

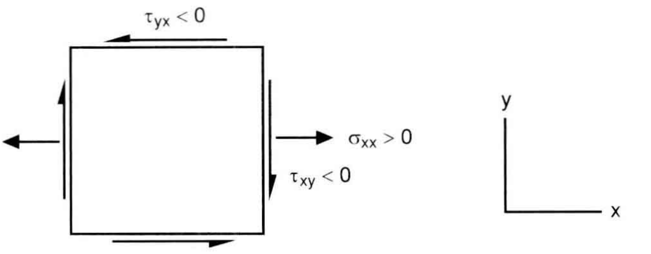


Figure 1.3. The normal stress, σ_{xx}, is positive because the direction of the force F_x and the plane A_x are either both positive (right) or both negative (left). The shear stress, τ_{xy} and τ_{yx}, are negative because the direction of the force and the normal to the plane have opposite signs.

positive and the x-direction force is negative, and on the bottom surface the x-direction force is positive and the normal to the area, y is negative. Similarly, τ_{xy} is negative.

Pairs of shear stress terms with reversed subscripts are always equal. A moment balance requires that $\tau_{ij} = \tau_{ji}$. If they were not, the element would rotate (Figure 1.4). For example, $\tau_{yx} = \tau_{xy}$. Therefore, we can write in general that $\Sigma M_A = \tau_{yx} = \tau_{xy} = 0$, so

$$\sigma_{ij} = \sigma_{ji}, \quad \text{or} \quad \tau_{ij} = \tau_{ji}. \tag{1.5}$$

This makes the stress tensor matrix symmetric about the diagonal.

Transformation of Axes

Frequently, it is useful to change the axis system on which a stress state is expressed. For example, we may want to find the shear stress on a inclined plane from the external stresses. Another example is finding the normal stress across a glued joint in a tube subjected to tension and torsion. In general, a stress state expressed along one set of orthogonal axes (e.g., m, n, and p) may be expressed along a different set of orthogonal axes (e.g., i, j, and k).

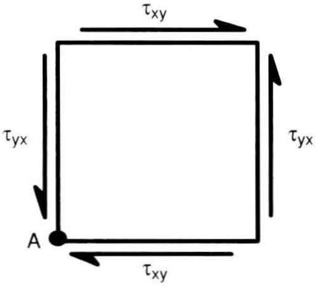
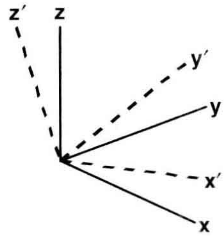


Figure 1.4. An infinitesimal element under shear stresses, τ_{xy} and τ_{yx}. A moment balance about A requires that $\tau_{xy} = \tau_{yx}$.

Figure 1.5. Two orthogonal coordinate systems, (x, y, z) and (x', y', z'). The stress state may be expressed in terms of either.



The general form of the transformation is

$$\sigma_{ij} = \sum_{n=1}^{3}\sum_{m=1}^{3} \ell_{im}\ell_{jn}\sigma_{mn}. \tag{1.6}$$

The term ℓ_{im} is the cosine of the angle between the i and m axes, and ℓ_{jn} is the cosine of the angle between the j and n axes. The summations are over the three possible values of m and n, namely m, n, and p. This is often written as

$$\sigma_{ij} = \ell_{im}\ell_{jn}\sigma_{mn} \tag{1.7}$$

with the summation implied. The stresses in the (x, y, z) coordinate system in Figure 1.5 may be transformed onto the (x', y', z') coordinate system by

$$\begin{aligned}\sigma_{x'x'} &= \ell_{x'x}\ell_{x'x}\sigma_{xx} + \ell_{x'y}\ell_{x'x}\sigma_{yx} + \ell_{x'z}\ell_{x'x}\sigma_{zx}\\ &\quad + \ell_{x'x}\ell_{x'y}\sigma_{xy} + \ell_{x'y}\ell_{x'y}\sigma_{yy} + \ell_{x'z}\ell_{x'y}\sigma_{zy}\\ &\quad + \ell_{x'x}\ell_{x'z}\sigma_{xz} + \ell_{x'y}\ell_{x'z}\sigma_{yz} + \ell_{x'z}\ell_{x'z}\sigma_{zz}\end{aligned} \tag{1.8a}$$

and

$$\begin{aligned}\sigma_{y'y'} &= \ell_{x'x}\ell_{y'x}\sigma_{xx} + \ell_{x'y}\ell_{y'x}\sigma_{yx} + \ell_{x'z}\ell_{y'x}\sigma_{zx}\\ &\quad + \ell_{x'x}\ell_{y'y}\sigma_{xy} + \ell_{x'y}\ell_{y'y}\sigma_{yy} + \ell_{x'z}\ell_{y'y}\sigma_{zy}\\ &\quad + \ell_{x'x}\ell_{y'z}\sigma_{xz} + \ell_{x'y}\ell_{y'z}\sigma_{yz} + \ell_{x'z}\ell_{y'z}\sigma_{zz}.\end{aligned} \tag{1.8b}$$

These equations may be simplified with the notation in Equation 1.3 using Equation 1.5,

$$\begin{aligned}\sigma_{x'} &= \ell_{x'x}^2\sigma_x + \ell_{x'y}^2\sigma_y + \ell_{x'z}^2\sigma_z\\ &\quad + 2\ell_{x'y}\ell_{x'z}\tau_{yz} + 2\ell_{x'z}\ell_{x'x}\tau_{zx} + 2\ell_{x'x}\ell_{x'y}\tau_{xy}\end{aligned} \tag{1.9a}$$

and

$$\begin{aligned}\tau_{x'z'} &= \ell_{x'x}\ell_{y'x}\sigma_{xx} + \ell_{x'y}\ell_{y'y}\sigma_{yy} + \ell_{x'z}\ell_{y'z}\sigma_{zz}\\ &\quad + (\ell_{x'y}\ell 1_{y'z} + \ell_{x'z}\ell_{y'y})\tau_{yz} + (\ell_{x'z}\ell_{y'x} + \ell_{x'x}\ell_{y'z})\tau_{zx}\\ &\quad + (\ell_{x'x}\ell_{y'y} + \ell_{x'y}\ell_{y'x})\tau_{xy}.\end{aligned} \tag{1.9b}$$

Now reconsider the transformation in Figure 1.2. Using equations 1.9a and 19b with σ_{yy} as the only finite term on the (x, y, z) axis system,

$$\sigma_{y'} = \ell_{y'y}^2\sigma_{yy} = \sigma_y\cos^2\theta \quad \text{and} \quad \tau_{x'y'} = \ell_{x'y}\ell_{y'y}\sigma_{yy} = \sigma_y\cos\theta\sin\theta \tag{1.10}$$

in agreement with Equations 1.4a and 1.4b.

Principal Stresses

It is always possible to find a set of axes (1, 2, 3) along which the shear stress components vanish. In this case the normal stresses, σ_1, σ_2, and σ_3, are called *principal stresses* and the 1, 2, and 3 axes are the *principal stress axes*. The magnitudes of the principal stresses, σ_p, are the three roots of

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0, \tag{1.11}$$

where

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz}, \\ I_2 &= \sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2 - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} - \sigma_{xx}\sigma_{yy} \\ I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{yz}\sigma_{zx}\sigma_{xy} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2. \end{aligned} \tag{1.12}$$

The first invariant is $I_1 = -p/3$, where p is the pressure. I_1, I_2, and I_3 are independent of the orientation of the axes and are therefore called *stress invariants*. In terms of the principal stresses, the invariants are

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= -\sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11} - \sigma_{11}\sigma_{22} \\ I_3 &= \sigma_{11}\sigma_{22}\sigma_{33}. \end{aligned} \tag{1.13}$$

EXAMPLE PROBLEM #1.1: Find the principal stresses in a body under the stress state, $\sigma_x = 10$, $\sigma_y = 8$, $\sigma_z = -5$, $\tau_{yz} = \tau_{zy} = 5$, $\tau_{zx} = \tau_{xz} = -4$, and $\tau_{xy} = \tau_{yx} = -8$, where all stresses are in MPa.

Solution: Using Equation 1.13, $I_1 = 10 + 8 - 5 = 13$, $I_2 = 5^2 + (-4)^2 + (-8)^2 - 8(-5) - (-5)10 - 10.8 = 115$, $I_3 = 10.8(-5) + 2.5(-4)(-8) - 10.5^2 - 8(-4)^2 - (-5)(-8)^2 = -138$ MPa.

Solving Equation 1.11, $\sigma_p^3 - 13\sigma_p^2 - 115\sigma_p + 138 = 0$, $\sigma_p = 1.079, 18.72, -6.82$ MPa.

Mohr's Stress Circles

In the special case where there are no shear stresses acting on one of the reference planes (e.g., $\tau_{zy} = \tau_{zx} = 0$), the normal to that plane, z, is a principal stress direction and the other two principal stress directions lie in the plane. This is illustrated in Figure 1.6. For these conditions, $\ell_{x'z} = \ell_{y'z} = 0$, $\tau_{zy} = \tau_{zx} = 0$, $\ell_{x'x} = \ell_{y'y} = \cos\phi$, and $\ell_{x'y} = -\ell_{y'x} = \sin\phi$. The variation of the shear stress component $\tau_{x'y'}$ can be found by substituting these conditions into the stress transformation Equation (1.8b). Substituting $\ell_{x'z} = \ell_{y'z} = 0$,

$$\tau_{x'y'} = \cos\phi\sin\phi(-\sigma_{xx} + \sigma_{yy}) + (\cos^2\phi - \sin^2\phi)\tau_{xy}. \tag{1.14a}$$

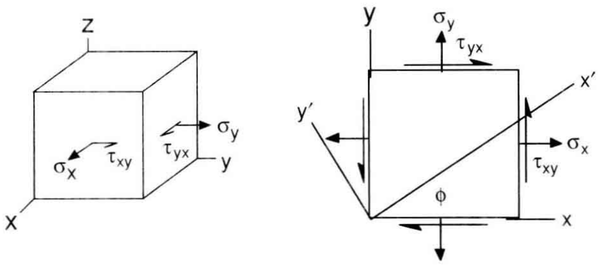


Figure 1.6. Stress state to which Mohr's circle treatment applies. Two shear stresses, τ_{yz} and τ_{zx}, are zero.

Similar substitution into the expressions for $\sigma_{x'}$ and $\sigma_{y'}$ results in

$$\sigma_{x'} = \cos^2\phi\sigma_x + \sin^2\phi\sigma_y + 2\cos\phi\sin\phi\tau_{xy} \tag{1.14b}$$

and

$$\sigma_{y'} = \sin^2\phi\sigma_x + \cos^2\phi\sigma_y + 2\cos\phi\sin\phi\tau_{xy}. \tag{1.14c}$$

These can be simplified by substituting the trigonometric identities $\sin 2\phi = 2\sin\phi\cos\phi$ and $\cos 2\phi = \cos^2\phi - \sin\phi$,

$$\tau_{x'y'} = -[(\sigma_x - \sigma_y)/2]\sin 2\phi + \tau_{xy}\cos 2\phi \tag{1.15a}$$

$$\sigma_{x'} = (\sigma_x + \sigma_y)/2 + [\sigma_x - \sigma_y)/2]\cos 2\phi\tau_{xy}\sin 2\phi \tag{1.15b}$$

and

$$\sigma_{y'} = (\sigma_x + \sigma_y)/2 - [(\sigma_x - \sigma_y)/2]\cos 2\phi + \tau_{xy}\sin 2\phi. \tag{1.15c}$$

Setting $\tau_{x'y'} = 0$ in Equation 1.15a, ϕ becomes the angle, θ, between the principal stresses axes and the x and y axes (see Figure 1.7):

$$\tau_{x'y'} = 0 = \sin 2\theta(\sigma_x - \sigma_y)/2 + \cos 2\theta\tau_{xy} \quad \text{or} \quad \tan 2\theta = \tau_{xy}/[(\sigma_x - \sigma_y)/2]. \tag{1.16}$$

The principal stresses σ_1 and σ_2 are the values of $\sigma_{x'}$ and $\sigma_{y'}$ for this value of ϕ,

$$\begin{aligned} \sigma_{1,2} &= (\sigma_x + \sigma_y)/2 \pm [\sigma_x - \sigma_y)/2]\cos 2\theta + \tau_{xy}\sin 2\theta \quad \text{or} \\ \sigma_{1,2} &= (\sigma_x + \sigma_y)/2 \pm (1/2)\left[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\right]^{1/2} \end{aligned} \tag{1.17}$$

A Mohr's circle diagram is a graphical representation of Equations 1.16 and 1.17. It plots as a circle with a radius $(\sigma_1 - \sigma_2)/2$ centered at

$$(\sigma_1 + \sigma_2)/2 = (\sigma_x + \sigma_y)/2 \tag{1.17a}$$

as shown in Figure 1.7. The normal stress components, σ, are represented on the ordinate and the shear stress components, τ, on the abscissa. Consider the