南京航空航天大学
研究生系列精品教材

Introduction to Matrix Theory

(矩阵论引论)

Zhengsheng Wang

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Preface

About matrix theory

The study of matrix theory copies a singular place within mathematicians. No applied mathematician can be properly trained without some basic understanding of matrix theory and linear algebra. It is still an area of active research, and it is used by every mathematician and by many scientists working in various specialities. Here we list several examples to illustrate its versatility.

(1) Scientific computing libraries begin growing around matrix computation. As a matter of fact, the discretization of partial differential operators is an endless source of linear finite-dimensional problems.

(2) At a discrete level, the maximum principle is related to nonnegative matrices.

(3) Control theory and stabilization of systems with finitely many degrees of freedom involve special analysis of matrices.

(4) The discrete Fourier transform(FT), including the fast Fourier transform(FFT), makes use of Toeplitz matrices.

(5) Statistics is widely based on correlation matrices.

(6) The generalized inverse is involved in least-squares approximation.

(7) Symmetric matrices are inertia, deformation, or viscous tensors in continuum mechanics.

(8) Markov process involves stochastic or bistochastic matrices.

(9) Graphs can be described in a useful way by square matrices.

(10) Quantum chemistry is intimately related to matrix groups and their representations.

About this book

Our goal is to provide a textbook for a course in matrix theory and methods accessible to advanced undergraduate and beginning graduate students. Through the course, students learn, practice, and master basic matrix results and techniques that are very useful for applications in various fields such as mathematics, statistics, physics, computer science, and engineering, etc. This book is an attempt to provide some of the required knowledge and understanding. It is written in a spirit that considers matrix theory not merely as a tool for solving applied problems but also as a challenging and rewarding part of mathematics.

This book has evolved over six years from the author's lecture notes of the graduate level course "matrix theory" for international students at Nanjing University of Aeronautics and Astronautics(NUAA). The book can serve as textbook or reference for graduate students in Master or PhD degree. The only prerequisites are a decent background in elementary linear algebra and calculus. This book can also be used as a reference for researchers and engineers. In order to make the book accessible not only to mathematicians but also to scientists and engineers, I have planned it to be as self-contained as possible.

This book is organized into nine chapters and consists of the following topics.

(1) Review and Miscellanea: Basic Concepts in Linear Algebra.

(2) Linear Space and Inner-product Space.

(3) Linear Transformation.

(4) Jordan Canonical Form.

(5) Matrix Factorization.

(6) Hermitian Matrix and Positive Definite Matrix.

(7) Matrix Norm and Matrix Analysis.

(8) Generalized Inverse Matrix.

(9) An Introduction to MATLAB.

The author has benefited from numerous books and journals. This book would not exit without the earlier works of a great number of authors. The references at the end of the book are intended as a possible guide to some of the literature covering the topics of the individual chapters more exhaustively.

The author would appreciate any comments, suggestions, questions, criticisms, or corrections that readers may take the trouble of communicating to me. Readers are welcome to communicate me via e-mail: wangzhengsheng@nuaa.edu.cn.

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I gratefully acknowledge a number of professors and friends for many valuable suggestions and helpful remarks that both greatly improved the quality of the book and have been improving my research work in matrix theory and computation. Among them are Hua Dai (NUAA), Zhong-zhi Bai (Institute of Computational Mathematics and Scientific/Engineering Computing of Chinese Academy of Sciences) and Lothar Reichel (Kent State University, USA).

I am thankful to my colleagues and my students who have had to put up with my talking to them so often about matrices and their constant interest and useful comments. I am also grateful to Science Press staff members who handled this book,especially editor Zhongxing Zhang, the NUAA Graduate School staff members for help with publishing this book. I also thank College of Science of NUAA for providing released time for me to work on the project.

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Finally, I would also like to express my appreciation to my family.

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NUAA, Nanjing
March 2015

Frequently Used Notations and Terminology

$\mathcal{C}$ -All complex numbers

$\mathcal{R}$ -All real numbers

$\mathcal{C}^n$ -All complex vectors of order n

$\mathcal{R}^n$ -All real vectors of order n

$\mathcal{C}^{m\times n}$ -All $m \times n$ complex matrices

$\mathcal{R}^{m\times n}$ -All $m \times n$ real matrices

$dim(V)$ -Dimension of vector space V

$A = (a_{ij})$ -Matrix A with (i, j)-entry a_{ij}

I -Identity matrix

A^T (A^H) -Transpose (Conjugate transpose) of matrix A

A^{-1} -Inverse of matrix A

$Rank(A)$ -Rank of matrix A

$|A|$ or $det(A)$ -Determinant of matrix A

$tr(A)$ -Trace of matrix A

$\|\cdot\|$ -Norm of a vector or matrix

(u, v) -Inner product of vectors u and v

$\rho(A)$ -Spectral radius of matrix A

$\sigma_{max}(A)$ -Largest singular value (spectral norm) of matrix A

$\lambda_{max}(A)$ -Largest eigenvalue of matrix A

An $n \times n$ matrix A is said to be

(1) Hermitian if $A^H = A$

(2) normal if $A^H A = AA^H$

(3) unitary if $A^H A = AA^H = I$, i.e., $A^{-1} = A^H$

(4) positive semidefinite if $x^H Ax \geqslant 0$ for all vectors $x \in C^n$

(5) similar to B if $P^{-1}AP = B$ for some invertible matrix P

(6) unitary similar to B if $U^H AU = B$ for some unitary matrix U

(7) diagonalizable if $P^{-1}AP$ is diagonal for some invertible matrix P

(8) upper-triangular if all entries below the main diagonal are zero

(9) λ is an eigenvalue of $A \in \mathcal{C}^{n\times n}$ if $Ax = \lambda x$ for some nonzero vector $x \in \mathcal{C}^n$

Contents

Chapter 1

Review and Miscellanea: Basic Concepts in Linear Algebra

We briefly review, mostly without proof, the basic concepts and results taught in an elementary linear algebra course.

1.1 Matrix Concept and Special Matrices

The term of *matrix* was first introduced by the British mathematician James Joseph Sylvester in 1890. The word "matrix" is derived from the Indo-European root mater, meaning "mother". Matrices are indeed the core of linear algebra.

Firstly, we see the following two numbers tables(Table 1.1 and Table 1.2) in real life.

Table 1.1 The price table of four kinds of cans in three supermarkets

	P1	P2	P3	P4
S1	17	7	11	21
S2	15	9	13	19
S3	18	8	15	19

We can write it in brief in the form

$$\begin{bmatrix} 17 & 7 & 11 & 21 \\ 15 & 9 & 13 & 19 \\ 18 & 8 & 15 & 19 \end{bmatrix}_{3\times 4}$$

Table 1.2 The distances of three cities in China (km)

	Beijing	Nanjing	Shanghai	Hongkong
Beijing	0	900	1050	1980
Nanjing	900	0	300	1140
Shanghai	1050	300	0	1120
Hongkong	1980	1140	1120	0

We can write it in the brief form

$$\begin{bmatrix} 0 & 900 & 1050 & 1980 \\ 900 & 0 & 300 & 1140 \\ 1050 & 300 & 0 & 1120 \\ 1980 & 1140 & 1120 & 0 \end{bmatrix}_{4\times 4}$$

It is symmetric.

Definition 1.1.1(Matrix) An array of numbers (or symbols) in m rows and n columns is called an $m \times n$ matrix.

The notation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or

$$A = (a_{ij})_{m \times n}$$

Usually, we denote

$$A \in \mathcal{R}^{m \times n}, \quad A \in \mathcal{C}^{m \times n}$$

where $\mathcal{R}^{m \times n}$ and $\mathcal{C}^{m \times n}$ denote the set of $m \times n$ real and complex matrices.

The entry that appears at the intersection of the ith row and jth column is called the (i, j) entry.

An $n \times n$ matrix is called a square matrix of order n. If the number of rows is not equal to the number of columns, the the matrix is called rectangular.

A matrix obtained from a given matrix A by deleting some (but of course not all) of its rows and columns is called the submatrix of A.

Example 1.1.1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Here, A is a 4×3 rectangular matrix, B is a square matrix of order 3, and B is a submatrix of A.

Definition 1.1.2 (Equality of matrices) Let $A = (a_{ij})_{m \times n}, B = (b_{ij})_{p \times q}$. Then A is said to be equal to B if

a) A and B have the same number of roes and the same number of columns, that is, $m = p, n = q$, and

b) the corresponding entries of A and B are equal, that is, $a_{ij} = b_{ij}$ for each pair of subscripts (i, j).

Here some special matrices which have special structure or/and special properties are introduced as follows.

Definition 1.1.3 (Zero matrix) The $m \times n$ matrix in which every entry is 0 is called the $m \times n$ zero matrix and is denoted by $0_{m \times n}$ or simply 0.

Note that

$$A + (-A) = A - A = 0, \quad A + 0 = A$$

Definition 1.1.4 (Diagonal matrix) An $n \times n$ matrix is called a diagonal matrix if each entry that is not on the diagonal is 0.

Example 1.1.2

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Definition 1.1.5 (Identity matrix) An $n \times n$ matrix is called an identity matrix if it is a diagonal matrix and all the entries on the diagonal are equal to 1.

Example 1.1.3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Definition 1.1.6 (Upper (Lower) triangular matrix) An $n \times n$ matrix is called an upper (lower) triangular matrix if all the entries below (above) the diagonal are zero.

Example 1.1.4

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 9 & 1 \end{bmatrix}_{3\times 3}, \quad U = \begin{bmatrix} 3 & 5 & 7 \\ 0 & 5 & 10 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

1.2 Matrix Algebra

Definition 1.2.1(Scalar multiplication) Let α be a number, and let $A = (a_{ij})_{m\times n}$ be a matrix. Then

$$\alpha A = A\alpha = (\alpha a_{ij})_{m\times n}$$

This means that αA is the matrix obtained by multiplying each entry of A by the same number α.

Example 1.2.1

$$3\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{bmatrix}$$

Definition 1.2.2 (Addition of matrices) Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be $m \times n$ matrices. Then

$$A + B = (c_{ij})_{m\times n}$$

where $c_{ij} = a_{ij} + b_{ij}$ for each pair (i, j).

Example 1.2.2

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 2 & -5 & 8 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 6 & 0 & 14 \\ 8 & 8 & 7 \end{bmatrix}$$

The negative of A is written as $-A$.

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 6 & 0 & 14 \\ 8 & 8 & 7 \end{bmatrix}, \quad -A = \begin{bmatrix} -1 & -1 & -4 \\ -6 & 0 & -14 \\ -8 & -8 & -7 \end{bmatrix}$$

Note that matrix addition is both commutative and associative, that is the theorem as follows.

Theorem 1.2.1 Let A, B, C be $m \times n$ matrices. Then
a) $A + B = B + A$ (commutative law of addition),
b) $(A + B) + C = A + (B + C)$ (associative law of addition).

Definition 1.2.3 (Matrix multiplication) If $A = (a_{ij})_{m\times n}, B = (b_{ij})_{n\times r}$, then the product $AB = C = (c_{ij})_{m\times r}$ matrix whose entries are defined by

$$c_{ij} = a(i,:)b(:,j) = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example 1.2.3

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$$

then

$$C = AB = \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix}$$

Matrix multiplication obeys many of the familiar laws of arithmetic apart from the commutative law.

Theorem 1.2.2 Suppose $A_{m\times n}, B_{n\times P}, C_{p\times q}$.
1) $(AB)C = A(BC)$;
2) $t(AB) = (tA)B = A(tB), A(-B) = (-A)B = -(AB)$;
3) $(A + B)C = AC + BC$;
4) $D(A + B) = DA + DB$.

In general, $AB \neq BA$. If $AB = BA$, we call that A commutes with B.

Example 1.2.4 (Application: A simple model for marital status computations) In a cetain town, 30 percent of the married women get divorced each year and 20 percent of the single women get married each year. There are 8000 married women and 2000 single women. Assuming that the total population of women remains constant, how many married women and how many single women will there be after 1 year? After 2 years?

Solution Form a matrix A as follows. The entries in the first row of A will be the percent of married and single women, respectively, that are married after 1 year. The entries in the second row will be the percent of women who are single after 1 year. Thus

$$A = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix}$$

If we let $x = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$, the number of married and single women after 1 year can be computed by multiplying A times x.

$$Ax = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

After 1 year there will be 6000 married women and 4000 single women. To find the number of married women and single women after 2 years, compute

$$A^2x = A(Ax) = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} \begin{bmatrix} 6000 \\ 4000 \end{bmatrix} = \begin{bmatrix} 5000 \\ 5000 \end{bmatrix}$$

After 2 years, half of the women will be married and half will be single. In general, the number of married and single women after n years can be determined by computing $A^n x$.

Definition 1.2.4 (Transpose of a matrix) Let $A = (a_{ij}) \in \mathcal{C}^{m \times n}$ be an $m \times n$ matrix. Then the transpose of A, written A^T, is the $n \times m$ matrix whose (i, j) entry is a_{ji} for all i, j. In other words, the ith row of A^T is the ith columns of A for all i.

Definition 1.2.5 (Symmetric matrix) An $n \times n$ matrix A is called symmetric if $A^T = A$. In other words, $A = (a_{ij})$ is symmetric if $a_{ij} = a_{ji}$ for all i, j.

For example, the following matrix is symmetric.

Example 1.2.5

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Definition 1.2.6 (Conjugate transpose of a matrix) Let $A = (a_{ij}) \in \mathcal{C}^{m \times n}$ be an $m \times n$ matrix. Then the conjugate transpose of A, written $A^H (A^*)$, is the $n \times m$ matrix whose (i, j) entry is $\overline{a_{ji}}$ for all i, j.

Example 1.2.6

$$A = \begin{bmatrix} 1 & 2-i & 3 \\ 4 & 5i & 6 \\ 7+8i & 8 & 9 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 & 7+8i \\ 2-i & 5i & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$A^H = \begin{bmatrix} 1 & 4 & 7-8i \\ 2+i & -5i & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Definition 1.2.7 (Trace of matrix) The trace of an $n \times n$ matrix $A = (a_{ij})$ is defined to be

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Note that

$$tr(A + B) = tr(A) + tr(B), \quad tr(AB) = tr(BA)$$

Example 1.2.7

$$tr(A) = tr\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 15$$

Definition 1.2.8 (Determinant of 1×1 and 2×2 matrices)

$$det(A) = |A| = |\begin{bmatrix} a_{11} \end{bmatrix}| = a_{11}$$

$$det(A) = |A| = \left|\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right| = a_{11}a_{22} - a_{12}a_{21}$$

Example 1.2.8

$$det(A) = |A| = |\begin{bmatrix} 3 \end{bmatrix}| = 3$$

$$\left|\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right| = 1 \times 4 - 2 \times 3 = -2$$

Definition 1.2.9 (Laplace expansion) Assume that the determinant function has been defined for matrices of size $(n-1)\times(n-1)$. Then $det(A)$ is defined by the so called first-row Laplace expansion:

$$\begin{aligned} det(A) =& a_{11}M_{11}(A) - a_{12}M_{12}(A) + \cdots + (-1)^{1+n}M_{1n}(A) \\ =& \sum_{j=1}^{n}(-1)^{1+j}a_{1j}M_{1j}(A) \\ =& \sum_{j=1}^{n} a_{1j}C_{1j} \end{aligned}$$

Example 1.2.9 $A=[a_{ij}]$ is a 3×3 matrix, the Laplace expansion gives

$$\begin{aligned} det(A) &= a_{11}M_{11}(A) - a_{12}M_{12}(A) + a_{13}M_{13}(A) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Definition 1.2.10 (Minor) Let $M_{ij}(A)$ (or simply M_{ij} if there is no ambiguity) denote the determinant of the $(N-1)\times(N-1)$ submatrix of A formed by deleting the ith row and jth column of A. ($M_{ij}(A)$ is called the (i,j) minor of A or the minor of entry a_{12})

Definition 1.2.11 (Cofactors) The cofactors of entry a_{ij} is definition as

$$C_{ij} = (-1)^{i+j}M_{ij}(A)$$

Definition 1.2.12 (The adjoint of matrix) The adjoint of matrix A is defined as

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Definition 1.2.13 (Inverse of matrix (Nonsingular matrix)) A square matrix $A \in \mathcal{C}^{n\times n}$ is called nonsingular or invertible if there exists a matrix $B \in \mathcal{C}^{n\times n}$ such that

$$AB = I_n = BA$$

Any matrix B with the above property is called an inverse of A, denote $A^{-1} = B$. If A does not have an inverse, A is called singular.

Theorem 1.2.3 (Inverses are unique) If A has inverses B and C, then $B = C$.

Theorem 1.2.4 (sufficient and necessary condition of nonsingular) If A is nonsingular (or invertible) $\Leftrightarrow det(A) \neq 0$. In another word, a square matrix is invertible if and only if its determinant is nonzero. and

$$A^{-1} = \frac{1}{det(A)}adj(A)$$

Definition 1.2.14 (Elementary row operations on a matrix: elementary row matrices) There are three types, E_{ij}; $E_{i(t)}$; $E_{ij(t)}$, corresponding to the three kinds of elementary row operation:

1) $E_{ij}(i \neq j)$ is obtained from the identity matrix I_n by interchanging rows i and j;
2) $E_i(t)(t \neq 0)$ is obtained by multiplying the ith row of I_n by t;
3) $E_{ij}(t)(i \neq j)$ is obtained from I_n by adding t times the jth row of I_n to the ith row.

Example 1.2.10 $(n = 3)$

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{23}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The elementary row matrices have the following distinguishing property.

Theorem 1.2.5 If a matrix A is premultiplied by an elementary row matrix, the resulting matrix is the one obtained by performing the corresponding elementary row operation on A.

Example 1.2.11

$$E_{23} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \end{bmatrix}$$

Definition 1.2.15 (Row equivalent) Recall that A and B are row equivalent if B is obtained from A by a sequence of elementary row operations, denote $A \simeq B$.

Theorem 1.2.6 (Equivalent canonical form) Any matrix A is row equivalent to the following form

$$A \simeq \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

Definition 1.2.16 (Rank of matrix) Suppose the number of nonzero entries on the diagonal of A's equivalent canonical form be r, the rank of matrix A is said to be r, denoted $\text{Rank}(A) = r$.

Theorem 1.2.7 Let A be nonsingular $n \times n$ matrix. Then
(1) A is row equivalent to I_n;
(2) A is a product of elementary row matrices.

1.3 Eigenvalue and Eigenvector

What does eigenvalue mean?

In mathematics, eigenvalue and eigenvector are related concepts in the field of linear algebra and matrix theory. The word eigenvalue comes from the German word "eigenwert" where the prefix "eigen" means "characteristic" "innate" "distinct", "self" and "wert" means "value". Eigenvalues and eigenvectors are properties of a matrix. They give important information about the matrix. They have applications in many areas of engineering, such as vibration system, quantum mechanics, finance and so on.

But what the word means is not on your mind! You want to know why do I need to learn about eigenvalues and eigenvectors. Once I give you an example of the application of eigenvalues and eigenvectors, you will want to know how to find these eigenvalues and eigenvectors. That is the motive of this introduction of this section.