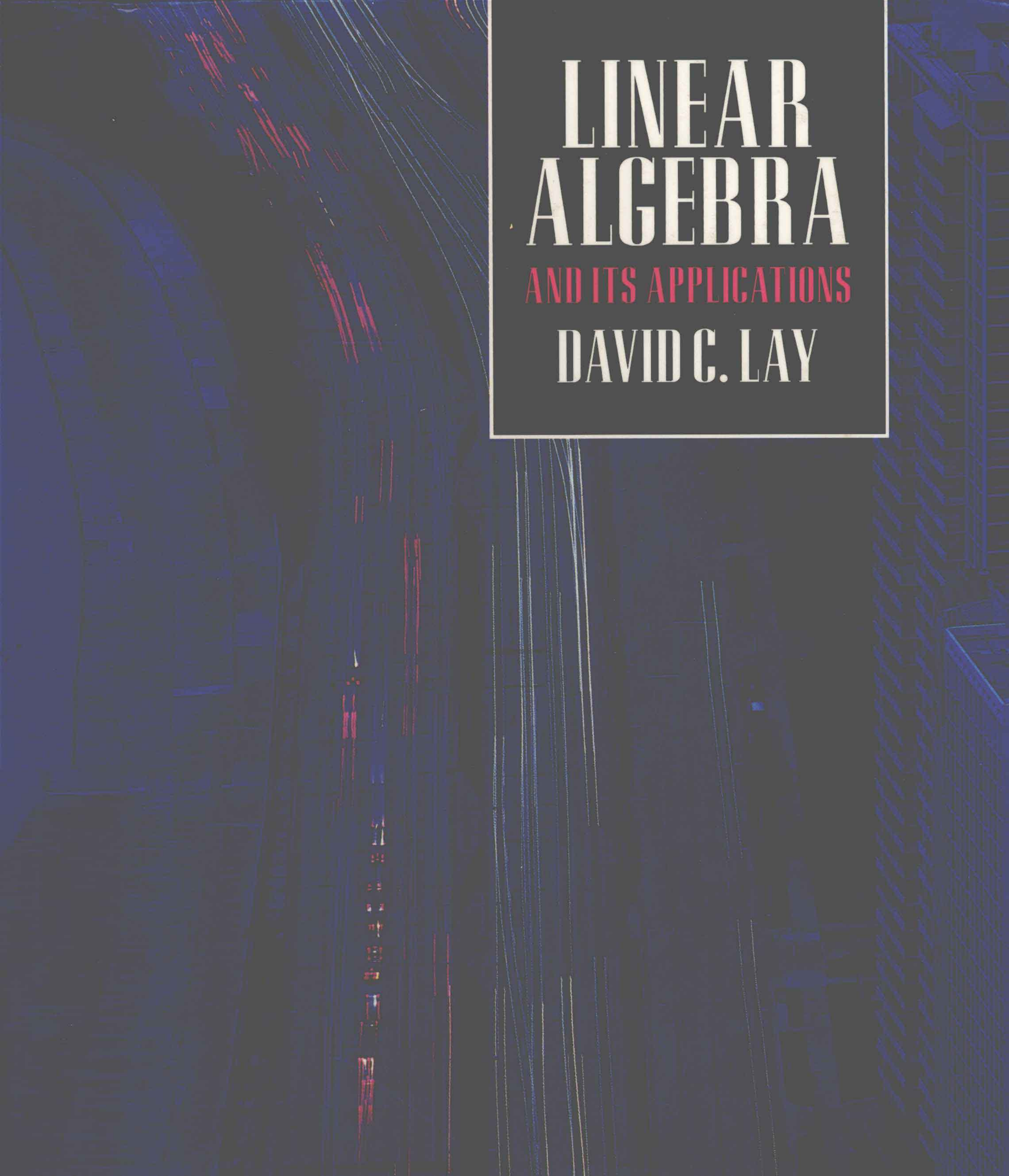
LINEAR ALGEBRA

AND ITS APPLICATIONS

DAVID C. LAY



LINEAR ALGEBRA and Its Applications

David C. Lay

University of Maryland

Addison-Wesley Publishing Company

Reading, Massachusetts • Menlo Park, California • New York
Don Mills, Ontario • Wokingham, England • Amsterdam • Bonn
Sydney • Singapore • Tokyo • Madrid • San Juan • Milan • Paris

Sponsoring Editor: Jerome Grant
Production Supervisor: Karen Wernholm/Mona Zeftel
Marketing Manager: Andrew Fisher
Development Editor: Lenore Parens
Copy Editor: Joyce Grandy
Cover Designer: Leslie Haimes
Manufacturing Manager: Roy Logan

Photo/illustration credits

Pages 95, 96 Courtesy of Ford Motor Company. Page 122 Copyright © Chips and Technologies, Inc., 1993. Reprinted with permission of the copyright owners. All rights reserved. Page 152 Photo by Bo Strain. Reprinted by permission of The University of North Carolina at Chapel Hill. Page 190 Courtesy of Rockwell International Corporation, Space Systems Division. Page 331 The Bettmann Archive. Page 332 From *North American Datum of 1983*, editor Charles Schwartz, National Geodetic Information Center. Page 379 Adapted from Goldstein/Lay/Schneider, *Calculus and Its Applications* 5/E, © 1990, p. 23. Reprinted by permission of Prentice-Hall, Inc., Englewood Cliffs, N.J. Page 403 Adapted with permission from Earth Observation Satellite Company, Lanham, MD USA. Page 404 GEOPIC® images courtesy of Earth Satellite Corporation, Rockville, MD.

Library of Congress Cataloging-in-Publication Data
Lay, David C.
Linear algebra and its applications / David C. Lay.
p. cm.
Includes index.
ISBN 0-201-52031-1 ISBN 0-201-52032-X (Instructor's Edition)
1. Algebras, Linear. I. Title.
QA184.L397 1994
512'.5—dc20 93-31730
CIP

Reprinted with corrections, January 1996.

5 6 7 8 9 10-CRW-96

Preface

This text provides a modern elementary introduction to linear algebra and some of its interesting applications, accessible to students with the maturity that should come from successfully completing two semesters of college-level mathematics, usually calculus.

The main goal of the text is to help students master the basic concepts and skills they will use later in their careers. The topics here follow the recommendations of the Linear Algebra Curriculum Study Group, which were based on a careful investigation of the real needs of the students and a consensus among professionals in many disciplines that use linear algebra. Hopefully, this course will be one of the most useful and interesting mathematics classes taken as an undergraduate.

DISTINCTIVE FEATURES

Early Introduction of Key Concepts

The text features a gradual but steady development of the subject—from simple ideas about systems of equations and their matrix representation to more challenging concepts in linear algebra. Each topic moves from elementary examples to general principles. Certain key ideas are introduced early in $\mathbb{R}^n$ and gradually examined from different points of view. Later generalizations of these concepts appear as natural extensions of familiar ideas, visualized through the geometric intuition developed in Chapter 2. A major achievement of the text, I believe, is that the level of difficulty for the students is fairly even throughout the course.

A Modern View of Matrix Multiplication

Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. This approach simplifies many arguments, and it permits vector space ideas to be tied into the study of linear systems. For instance, students can find spanning and linear independence difficult to understand, even in the context of $\mathbb{R}^n$. However, after years of experimentation, I have found that the initial contact with these ideas is easier when students have a proper understanding of matrix-vector multiplication, viewing $A\mathbf{x}$ as a linear combination of the columns of A.

Linear Transformations

Linear transformations form a "thread" that is woven into the fabric of the text. Their use enhances the geometric flavor of the text. In Chapter 2, for instance, linear transformations provide a dynamic and graphical view of matrix-vector multiplication. Then, linear transformations appear repeatedly as new concepts are discussed. This gradual and concrete approach eventually leads to a solid understanding of a fairly difficult yet essential topic.

Eigenvalues and Dynamical Systems

Too often, eigenvalues are treated hurriedly at the end of a linear algebra course. In this text, students study eigenvalues in both Chapters 6 and 8. Because this material is spread over several weeks, students have more time than usual to absorb and review these critical concepts. The discussion in Chapter 6 is motivated by and applied to discrete dynamical systems. I believe that the graphical descriptions of such systems, including those with complex eigenvalues, appear here for the first time in an elementary linear algebra text.

Orthogonality and Least-Squares Problems

These topics receive a more comprehensive treatment than is commonly found in beginning texts. The Linear Algebra Curriculum Study Group has emphasized the need for a substantial unit on orthogonality and least-squares problems, because orthogonality plays such an important role in computer calculations and numerical linear algebra and because inconsistent linear systems arise so often in practical work. The foundation for this material is laid in Sections 7.1 to 7.3. After that a variety of topics, such as the QR factorization, can be covered as time permits. General inner product spaces are treated in the last two sections of Chapter 7.

PEDAGOGICAL FEATURES

Applications

A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science, mathematics, physics, biology, economics, and statistics. Each chapter opens with an introductory vignette that sets the stage for some application of linear algebra and provides a motivation for developing the mathematics that follows. Later, the text returns to that application in a section near the end of the chapter. Shorter applications are scattered throughout the text, and some chapters have more than one application section at the end.

A Strong Geometric Emphasis

Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualize an idea. There are substantially more drawings here than usual, and some of the figures have never appeared before in a linear algebra text.

Examples

This text devotes a larger proportion of its expository material to examples than most linear algebra texts. There are more examples than one would ordinarily present in class. But because the examples are written carefully, with lots of detail, students can read them on their own.

Theorems and Proofs

Important results are stated as theorems. Other useful facts are displayed in blue-tinted boxes, for easy reference. These boxed facts are usually justified in an informal discussion, as are several of the theorems. In a few cases, the essential calculations of a proof are exhibited in a carefully chosen example. Most of the theorems, however, have formal proofs, written with the beginning student in mind. The proofs illustrate how various concepts and definitions are used in arguments, and they provide model arguments for students who are learning to construct proofs. A few routine verifications are saved for exercises, when they will benefit students. The only two central facts stated without proof concern cofactor expansions of a determinant and the orthogonal diagonalization of a symmetric matrix.

Practice Problems

One to four carefully selected problems appear just before each exercise set. Complete solutions follow the exercise set. These problems either focus on potential trouble spots in the exercise set or provide a "warm-up" to the exercises, and the solutions often contain helpful hints or warnings about the homework.

Exercises

The abundant supply of exercises ranges from routine computations to conceptual questions that require more thought. A good number of innovative questions pinpoint conceptual difficulties that I have found on student papers over the years. Each exercise set is carefully arranged, leading from one idea to the next, in the same general order as the text; homework assignments are readily available when only part of a section is discussed. A notable feature of the exercises is their numerical simplicity. Problems "unfold" quickly, so students spend little time on numerical calculations. The exercises concentrate on teaching understanding rather than mechanical calculations.

Writing Exercises

An ability to write coherent mathematical statements in English is essential for all students of linear algebra, not just those who may go to graduate school in mathematics. Engineers and scientists need to be able to communicate *why* something is true, to say precisely what they mean. For example, an engineer may need to justify to a superior why a new design will work, or a scientist may want to prepare a research grant proposal.

To develop this ability to write, I have included many exercises for which a written justification is part of the answer. Some questions are open-ended; some are true/false/justify. Conceptual exercises that require a short proof usually contain hints that help a student get started. Or, an exercise is broken down into simple steps. For all odd-numbered writing exercises, either a solution is included at the back of the text, or a hint is given and the solution is in the Study Guide, described below.

Computational Topics

The text stresses the impact of the computer on both the development and practice of linear algebra in science and engineering. Frequent "Numerical Notes" draw attention to issues in computing and distinguish between theoretical concepts, such as matrix inversion, and computer implementations such as LU factorizations. Forty-five exercises, in application sections and in two sections on iterative methods, require a computer or supercalculator for their solutions.

SUPPLEMENTS

Study Guide

I wrote this paperback student supplement to be an integral part of the course. It complements the text in several ways. Detailed solutions are given to every third odd problem: 1, 7, 13, . . . , and most key exercises are covered in this way. Instructors can refer students to these explanations whenever the class discussion time is limited. Also, solutions are given to *every* odd-numbered writing exercise whenever the text's answer is only a "Hint."

The Study Guide, however, is much more than a solutions manual. It contains numerous warnings about common student errors, along with other hints and suggestions for studying, written the way I talk to my own students; it provides a separate glossary for each chapter, to help students prepare for exams; and it includes explanations of how various proofs are designed and logically arranged, to assist those students who are learning to construct proofs. Also, since more and more students are using MATLAB each year, the Study Guide describes the appropriate MATLAB commands when they are first needed.

MATLAB Data Bank

Data for over 700 numerical exercises are available in MATLAB readable M-files, to encourage the use of MATLAB. Since these files eliminate the need to enter and check data, their use can save time on homework, even though most of the exercises involve minimal calculations. The M-files are on a disk supplied with the Instructor's Edition; also included are special M-files that make MATLAB more useful in a teaching environment. Students can obtain a free copy of all the M-files directly from The MathWorks, which produces MATLAB, by mailing a card attached to the Study Guide.

Instructor's Edition

For the convenience of instructors, this special edition includes brief answers to all exercises. A *Note to the Instructor* at the beginning of the text provides a commentary on the design and organization of the text, to help instructors plan their course. Suggestions for course syllabi are given, based on a topical division of the text into 26 core sections, 13 supplementary sections, and 11 applications sections.

ACKNOWLEDGMENTS

I am indeed grateful to many groups of people who have helped me over the years with various aspects of this book.

First, I sincerely thank Israel Gohberg and Robert Ellis for the nearly fifteen years of research collaboration in linear algebra, which has so greatly shaped my view of linear algebra. Because our research has had connections with problems in signal processing and engineering control, I have had opportunities to see firsthand the importance of linear algebra in science and engineering.

Next, I express my appreciation to my other good friends and colleagues at College Park, with whom I have had many fruitful conversations about linear algebra during the past ten to fifteen years: Ellen Correl, Jerome Dancis, Seymour Goldberg, Richard Good, Denny Gulick, Garry Helzer, and also Paul Smith, who has helped me greatly with the various applications to statistics that are now in the text. I appreciate, too, the valuable suggestions received from Der-Chen Chang, Robert Ellis, and John Horváth, while teaching from my text several times in recent years.

I am indebted to the mathematics department at the University of Maryland for giving me opportunities to teach many small classes of linear algebra and then to try out my material several times in the large lecture format. Also, I am thankful for the comments of the hundreds of students who have used photocopied versions of the text over the past six years. Two such "students" were Professors Joann and Harvey Keselman, on sabbatical leave from the University of Manitoba, who audited my course and made numerous helpful suggestions, particularly about statistical applications.

It has been my privilege to work with David Carlson, Charles Johnson, and Duane Porter, on the Linear Algebra Curriculum Study Group. Their ideas about teaching linear algebra, along with contributions of the other members of the William and Mary Workshop, have influenced this text in several important ways.

For assistance with the chapter opening examples and subsequent discussions, I thank Professor Wassily Leontief, Institute for Economic Analysis,

New York University; Clopper Almon, University of Maryland; Edward F. Moylan, Ford Motor Company; Roland Lamberson, Humboldt State University; and Russell Hardie and Chris Peterson, both of Earth Satellite Corporation. Each person named was very generous with time, advice, and materials such as photographs and data. Also, I thank Michael Butler, of the College of the Redwoods, for suggesting the spotted owl research in California.

The text has been reviewed at several stages, and I sincerely thank the following faculty for their careful analyses and constructive suggestions:

Susan Jane Colley, Oberlin College
Murray Eisenberg, University of Massachusetts at Amherst
Laxmi N. Gupta, Rochester Institute of Technology
James F. Hurley, University of Connecticut
Mohammed Kazemi, University of North Carolina
Andrzej Kedzierawski, Delaware State College
Robert A. Maynard, Tidewater Community College
Chris Nieman, Belleville Area College
Michael A. Penna, Indiana University—Purdue University at Indianapolis
Stephen Pennell, University of Massachusetts at Lowell
Carlos A. Pereira, The King's College
John Rhodes, Bates College
James Rowell, University of Minnesota—Duluth
Joseph B. Stephen, Northern Illinois University
Mo Tavakoli, Chaffey College
W. Vance Underhill, East Texas State University
Monte J. Zerger, Adams State College

Over the past three years, I have been fortunate to have many faculty class-test the manuscript. Their comments have been most useful, and I am happy to express my appreciation to:

Wayne Barrett, Brigham Young University
Jane Day, San Jose State University
Diane Doyle, Adirondack Community College
Luz DeAlba, Drake University
Evarist Giné, University of Connecticut
William Hager, University of Florida
Gaston Hernandez, University of Connecticut
James F. Hurley, University of Connecticut
Jonathan King, University of Florida
Israel Koltracht, University of Connecticut
Walter B. Lowrie, University of Connecticut

Wolodymyr Madych, University of Connecticut
Michael Menaker, University of Maryland University College
Michael Neumann, University of Connecticut
Richard Orr, Rochester Institute of Technology
Richard Pfeifer, San Jose State University
Leiba Rodman, The College of William and Mary
James Rowell, University of Minnesota—Duluth
Jon Scott, Montgomery College (Takoma Park)
Rick Smith, University of Florida
Margaret Suchow, Adirondack Community College
Gilbert Sward, Montgomery College (Rockville)
Linda Valdes, San Jose State University

A number of persons checked the accuracy of the text, and I really appreciate their work. William Gragg, of the Naval Postgraduate School, read the numerical notes and related topics and offered many good suggestions. Jon Scott, of Montgomery College, and Richard Good, of the University of Maryland, read the galleys and checked the numerical calculations in the text and examples. Professor Good was also very helpful in his careful review of the glossary and the solutions to the writing exercises. Some of my best students in recent years checked and double checked all the answers in the final manuscript: Yougandh Chitre, Leigh Ann Stuart, and Jennifer Yuin, together with graduate students Jawad Al-Khal and Ashvina Kulkarni.

Finally, I sincerely thank the staff at Addison-Wesley for their superb work on the book, especially Jerome Grant, senior mathematics editor, who orchestrated the entire publishing effort, and Lenore Parens, developmental editor, who took the students' point of view when reading and rereading the entire manuscript. My thanks also go to the rest of the team at Addison-Wesley responsible for my book: Karen Wernholm and Mona Zeftel, senior production supervisors, Meredith Nightingale, art coordinator, Andy Fisher, marketing manager, and Jim Behnke, President of Addison-Wesley Higher Education Group.

David C. Lay

A Note to Students

This course is potentially the most interesting and valuable undergraduate mathematics course you will study. The following remarks offer some advice and information to help you master the material and enjoy the course.

In linear algebra, the *concepts* are as important as the *computations*. The simple numerical exercises that begin each exercise set only help you check your understanding of basic procedures. Later in your career, computers will do the calculations, but you will have to choose the calculations, know how to interpret the results, and then explain the results to other people. For this reason, many exercises in the text ask you to explain or justify your calculations. A written explanation is often required as part of the answer. For odd-numbered exercises, you will find either the desired explanation or at least a good hint. You must avoid the temptation to look at such answers until you have tried to write out the solution yourself. Otherwise, you are likely to think you understand something when in fact you do not.

To master the concepts of linear algebra, you will have to read and reread the text carefully. New terms are in boldface type, sometimes enclosed in a definition box. A glossary of terms is included at the end of the text. Important facts are stated as theorems or are enclosed in tinted boxes, for easy reference.

In a practical sense, linear algebra is a language. You must learn this language the same way you would study a foreign language—with daily work. Material presented in one section is not easily understood unless you have thoroughly studied the text and worked the exercises for the preceding sections. Keeping up with the course will save you lots of time and distress!

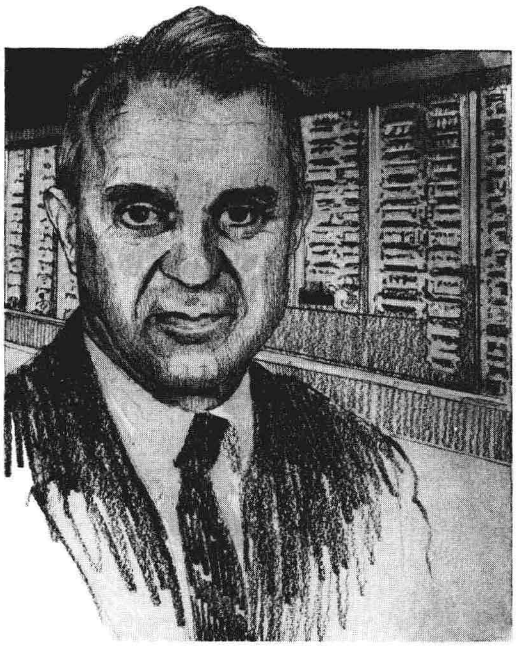
Study Guide

To help you succeed in this course, I have written a Study Guide to accompany the text. It contains detailed solutions to every third odd-numbered exercise, plus solutions to all odd-numbered exercises that only give a hint in the answer section. The Study Guide also provides warnings of common errors, helpful hints that call attention to key exercises and potential exam questions, and a separate glossary of terms for each chapter (invaluable when reviewing for an exam). Further, the Study Guide shows you how to use MATLAB (a computer program) to save hours of homework time. A postcard in the Study Guide entitles you to a free disk with data for over 700 exercises in the text. One simple command in MATLAB will retrieve all the data you need for a problem. If you have access to MATLAB (not included with the data disk), you will want to use this powerful aid.

Contents

1

Systems of Linear Equations



Introductory Example: Linear Models in Economics and Engineering

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained economic information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.3 and 3.7.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations.*

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration.* When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day.* The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.
- *Linear programming.* Many important management decisions today are made on the basis of linear programming models that utilize hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks.* Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. The software relies on linear algebra techniques and systems of linear equations.

A central concern of linear algebra is the study of systems of linear equations. Sections 1.1 and 1.2 present a systematic method for solving such systems. With only minor technical modifications, this method is the one used in most computer programs that solve systems of linear equations. Many of the examples and exercises here are so simple numerically that they could be solved by a variety of algebraic techniques. However, the method presented here must be mastered; it will be needed throughout the text.

1.1 INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they may be rearranged algebraically as in Eq. (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same set of variables, say, $x_1, \ldots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 \qquad - 4x_3 &= -7 \end{aligned} \tag{2}$$

A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by l_1 and l_2. A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both l_1 and l_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Fig. 1.

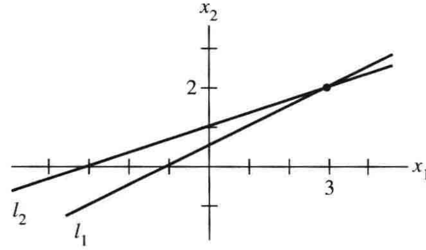


FIGURE 1 Exactly one solution.