B. Davies

Applied Mathematical Sciences 25

Integral Transforms and Their Applications

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Preface

This book is intended to serve as introductory and reference material for the application of integral transforms to a range of common mathematical problems. It has its immediate origin in lecture notes prepared for senior level courses at the Australian National University, although I owe a great deal to my colleague Barry Ninham, a matter to which I refer below. In preparing the notes for publication as a book, I have added a considerable amount of material additional to the lecture notes, with the intention of making the book more useful, particularly to the graduate student involved in the solution of mathematical problems in the physical, chemical, engineering and related sciences.

Any book is necessarily a statement of the author's viewpoint, and involves a number of compromises. My prime consideration has been to produce a work whose scope is selective rather than encyclopedic; consequently there are many facets of the subject which have been omitted--in not a few cases after a preliminary draft was written--because I believe that their inclusion would make the book too long. Some of the omitted material is outlined in various problems and should be useful in indicating possible approaches to certain problems. I have laid great stress on the use of complex variable techniques, an area of mathematics often unfashionable, but frequently of great power. I have been particularly severe in excising formal proofs, even though there is a considerable amount of "pure mathematics" associated with the understanding and use of generalized functions, another area of enormous utility in mathematics. Thus, for the formal aspects of the theory of integral transforms I must refer the reader to one of the many excellent books addressed to this area; I have chosen an approach which is more common in published research work in applications. I can only hope that the course which I have steered will be of great interest and help to students and research workers who wish to use integral transforms.

It was my priviledge as a student to attend lectures on mathematical physics by Professor Barry W. Ninham, now at this university. For several years it was his intention to publish a comprehensive volume on mathematical techniques in physics, and he prepared draft material on several important topics to this end. In 1972 we agreed to work on this project jointly, and continued to do so until 1975. During that period it became apparent that the size, and therefore cost, of such a large volume would be inappropriate to the current situation, and we decided to each publish a smaller book in our particular area of interest. I must record my gratitude to him for agreeing that one of his special interests--the use of the Mellin transform in asymptotics--

should be included in the present book. In addition there are numerous other debts which I owe to him for guidance and criticism.

References to sources of material have been made in two ways, since this is now a fairly old subject area. First, there is a selected bibliography of books, and I have referred, in various places, to those books which have been of particular assistance to me in preparing lectures or in pursuing research. Second, where a section is based directly on an original paper, the reference is given as a footnote. Apart from this, I have not burdened the reader with tedious lists of papers, especially as there are some comprehensive indexing and citation systems now available.

A great deal of the final preparation was done while I was a visitor at the Unilever Research Laboratories (UK) and at Liverpool University in 1975, and I must thank those establishments for their hospitality, and the Australian National University for the provision of study leave. Most of the typing and retyping of the manuscript has been done by Betty Hawkins of this department while the figures were prepared by Mrs. L. Wittig of the photographic services department, ANU. Timothy Lewis, of Applied Mathematics at Brown University, has proofread the manuscript and suggested a number of useful changes. To these people I express my gratitude and also to Professor Lawrence Sirovich for his encouragement and helpful suggestions. This book is dedicated to my respected friend and colleague, Barry Ninham.

Brian Davies

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Part I: The Laplace Transform

§1. DEFINITION AND ELEMENTARY PROPERTIES[1]

1.1. The Laplace Transform

Let $f(t)$ be an arbitrary function defined on the interval $0 \leq t < \infty$; then

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

is the Laplace transform of $f(t)$, provided that the integral exists. We shall confine our attention to functions $f(t)$ which are absolutely integrable on any interval $0 \leq t \leq a$, and for which $F(\alpha)$ exists for some real α. It may readily be shown that for such a function $F(p)$ is an analytic function of p for $\operatorname{Re}(p) > \alpha$, as follows. First note that the functions

$$\phi(p,T) = \int_0^T e^{-pt} f(t)\, dt \tag{2}$$

are analytic in p, and then that $\phi(p,T)$ converges uniformly to $F(p)$ in any bounded region of the p plane satisfying $\operatorname{Re}(p) > \alpha$, as $T \to \infty$. It follows from a standard theorem on uniform convergence[2] that $F(p)$ is analytic in the half-plane $\mathrm{Re}(p) > \alpha$.

As simple examples of Laplace transforms, we have

(i) Heaviside unit step function

$$h(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{3}$$

$$H(p) = \int_0^\infty e^{-pt}\, dt$$

$$= 1/p, \quad \mathrm{Re}(p) > 0 , \tag{4}$$

(ii)

$$f(t) = e^{i\omega t}, \quad \omega \text{ real} \tag{5}$$

$$F(p) = \int_0^\infty e^{-pt}\, e^{i\omega t}\, dt$$

$$= \frac{1}{p-i\omega}, \quad \mathrm{Re}(p) > 0, \tag{6}$$

(iii)

$$f(t) = t^\gamma\, e^{\beta t}, \quad \alpha \text{ real} > -1 \tag{7}$$

$$F(p) = \frac{\gamma!}{(p-\beta)^{\gamma+1}}, \quad \mathrm{Re}(p) > \mathrm{Re}(\beta). \tag{8}$$

An important feature of these examples, and indeed of many of the Laplace transforms which occur in applications, is that the analytic function defined by (1) in the half-plane $\mathrm{Re}(p) > \alpha$ can be analytically continued into the remainder of the plane once the singularity structure has been elucidated. Thus the functions defined by (4) and (6) exhibit only a simple pole; in the case of (8) there is a branch point at $p = \beta$ except for the special case that γ is an integer, when we get a pole.

1.2. Important Properties

There are a number of simple properties which are of recurring importance in the application of the Laplace transform to specific problems. In order to simplify somewhat the statement of these results, we introduce the notation

$$\mathscr{L}[f] = F(p) = \int_0^\infty e^{-pt} f(t)\, dt \tag{9}$$

which emphasizes the operator nature of the transform.

Linearity: If we consider the linear combination

$$f(t) = \sum_{k=1}^{n} a_k f_k(t) \tag{10}$$

where the a_k are arbitrary constants, then

$$\mathscr{L}[f] = \sum_{k=1}^{n} a_k \mathscr{L}[f_k] \; . \tag{11}$$

One immediate consequence of this is that if f depends on a variable x which is independent of t, we have

$$\mathscr{L}[\partial f/\partial x] = \partial \mathscr{L}[f]/\partial x, \tag{12}$$

$$\mathscr{L}\left[\int_a^b f\, dx\right] = \int_a^b \mathscr{L}[f]\, dx. \tag{13}$$

These results follow by trivial manipulation of the integrals in the half-space $Re(p) > \alpha$ in which all the integrals converge absolutely and uniformly (in x). But then they must also hold over the entire region of the complex p plane to which the transforms may be analytically continued.

Derivatives and Integrals: If we apply integration by parts to (1), we obtain