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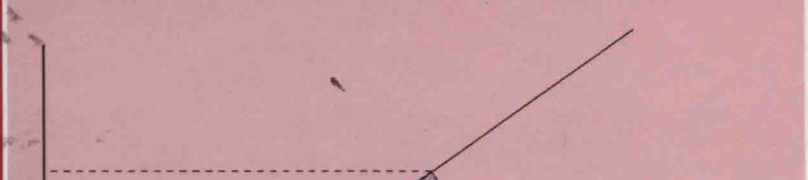
ISOLATED SINGULARITIES IN PARTIAL DIFFERENTIAL INEQUALITIES

Marius Ghergu and Steven D. Taliaferro



ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Isolated Singularities in Partial Differential Inequalities

MARIUS GHERGU
University College Dublin

STEVEN D. TALIAFERRO
Texas A & M University

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ISOLATED SINGULARITIES IN PARTIAL DIFFERENTIAL INEQUALITIES

In this monograph the authors present some powerful methods for dealing with singularities in elliptic and parabolic partial differential inequalities. Here, the authors take the unique approach of investigating differential inequalities rather than equations, the reason being that the simplest way to study an equation is often to study a corresponding inequality; for example, using sub- and superharmonic functions to study harmonic functions. Another unusual feature of the present book is that it is based on integral representation formulae and nonlinear potentials, which have not been widely investigated so far. This approach can also be used to tackle higher-order differential equations.

The book will appeal to graduate students interested in analysis, researchers in pure and applied mathematics, and engineers who work with partial differential equations. Readers will require only a basic knowledge of functional analysis, measure theory, and Sobolev spaces.

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Preface

This monograph is intended to present some powerful methods for dealing with singularities in elliptic and parabolic partial differential equations (PDEs). Singularities are a common feature of the qualitative side of mathematics; whether they appear in topology, differential geometry, or PDEs, the understanding of singularities always leads to a more detailed picture of the objects mathematics is dealing with.

This book invites the reader to a journey through modern techniques in dealing with singularities in PDEs and is addressed to researchers and graduate students having some deep interest in analysis. We believe this monograph presents some unique features in comparison with previous books on singular solutions. First, the emphasis throughout the book is on partial differential *inequalities* rather than PDEs. Inequalities are more robust than equations. Every equation is a special case of a non-strict inequality. Moreover, it is often the case that the easiest way to study an equation is to study a corresponding inequality. The simplest example in this respect is the use of sub- and superharmonic functions to study harmonic functions. Second, the present book brings a different approach based on integral representation formulae which has been little investigated so far. Such an approach is also suitable to tackle higher-order differential equations as we illustrate in Chapters 6 and 7.

Let us now provide the reader with an outline of our book.

Chapter 1: This chapter is an introductory part which presents the main tools of our approach. We pay particular attention to integral representations of solutions that exhibit isolated singularities. It is well known that if u is nonnegative and satisfies

$$-\Delta u \geq 0 \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n,\ n \geq 2,$$

then u can be written as

$$u(x) = m\Phi(|x|) + N(x) + h(x) \quad \text{in } B_1(0) \setminus \{0\}, \tag{1}$$

where h is a harmonic function in $B_1(0)$, $m \geq 0$ is a constant, Φ is a fundamental solution of $-\Delta$, and N is the Newtonian potential of $f := -\Delta u$. We derive similar representation formulae for the polyharmonic inequality

$$-\Delta^m u \geq 0 \quad \text{in } B_2(0) \setminus \{0\}$$

and the heat inequality

$$u_t - \Delta u \geq 0 \quad \text{in } B_2(0) \times (0, 1) \subset \mathbb{R}^n \times \mathbb{R}.$$

Chapter 2: Here we are concerned with the semilinear inequality

$$0 \leq -\Delta u \leq f(u) \quad \text{in } B_1(0) \setminus \{0\}. \tag{2}$$

Our aim is to find optimal conditions on f such that any positive solution u of (2) satisfies

$$u(x) = O(\varphi(|x|)) \quad \text{as } x \to 0 \tag{3}$$

for some continuous function $\varphi : (0, 1) \to (0, \infty)$.

Chapter 3: We continue the line of Chapter 2 where (2) is replaced with

$$au^p \leq -\Delta u \leq u^p$$

in various subsets of $\mathbb{R}^n$ ($n \geq 3$) where $a \in (0, 1)$ is a constant. We emphasize that the existence of a pointwise bound as in (3) is intimately related to the size of the constant a.

Chapter 4: This chapter deals with the inequality

$$-\Delta u - b(x)u \geq u^p \quad \text{in } \mathcal{D}'(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a cone-like domain with $0 \in \partial\Omega$, $p > 1$ and $b \in L^1_{loc}(\Omega)$ is a singular potential of Hardy-type. In the last part of this chapter we present the case where the solutions possess a higher dimensional singularity set.

Chapter 5: In this chapter we shall analyze elliptic inequalities of the type

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \geq u^p \quad \text{in } B_R(0) \setminus \{0\}.$$

Here we emphasize the role played by the matrix of coefficients $\{a_{i,j}(x)\}$ as well as the lower-order terms $\{b_i(x)\}$ which are assumed to be singular at the origin.

We observed an instability of the critical exponent p, a phenomenon that occurs even when the Laplace operator is perturbed by lower-order terms. To be more precise, let us consider the differential inequality

$$\Delta u + \beta \frac{x}{|x|^2} \cdot \nabla u \geq u^p \quad \text{in } B_R \setminus \{0\} \subset \mathbb{R}^n, n \geq 3. \tag{4}$$

The results in Chapter 5 allow us to deduce that (4) has singular solutions for all $-\infty < p < \infty$ if $\beta \leq 2 - n$ while if $\beta > 2 - n$ then singular solutions exist if and only if $-\infty < p < (n + \beta)/(n + \beta - 2)$.

Chapter 6: This part contains the study of singularities for higher-order inequalities of type

$$-\Delta^m u \geq 0 \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n. \tag{5}$$

We prove that any positive solution of (5) satisfies (3) if either m is even or $n < 2m$. We also consider the inequality (5) in an exterior domain.

Chapter 7: We shall focus on the inequality

$$-\Delta^2 u \geq f(u)$$

in exterior domains of $\mathbb{R}^n$. We give lower bounds on the growth of $f(s)$ at $s = 0$ and/or $s = \infty$ such that the above inequality has no C^4 positive solution in any exterior domain of $\mathbb{R}^n$. Since the maximum principle does not hold for the biharmonic operator, we adopt a different approach which relies on integral representation formulae established in Chapter 1.

Chapter 8: We analyze the parabolic inequality $0 \leq u_t - \Delta u \leq f(u)$ in various domains of $\mathbb{R}^n$. Our approach relies on deep estimates for the heat kernel as well as on various heat potential estimates which we provide in Appendices A and B.

Chapter 9: Chapter 9 of our book is devoted to the study of semilinear elliptic systems of inequalities

$$\begin{cases} 0 \leq -\Delta u \leq f(v) \\ 0 \leq -\Delta v \leq g(u) \end{cases} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 2,$$

where $f, g : (0, \infty) \to (0, \infty)$ are continuous functions. In using an integral representation such as (1) for nonnegative solutions u and v we end up with various nonlinear potentials of Havin-Maz'ya type which we discuss in Appendix C.

Chapter 10: Nonlocal problems have been intensely studied in the last decade. They arise in stochastic control theory, integro-differential equations, fluid mechanics (Boltzmann equation or even Navier–Stokes equation for a viscous fluid).

Chapter 10 of the book sheds some light on nonlocal semilinear systems. Our presentation is motivated by the double inequality

$$0 \leq -\Delta u \leq \left(\frac{1}{|x|^\alpha} * u \right)^p \quad \text{in } B_1(0) \setminus \{0\}, \ p \geq 0,$$

which, for $p = 1$ represents the degenerate Choquard inequality. We are able to relate our study with the Hardy–Littlewood–Sobolev inequality.

Chapter 11: Motivated by the results in Chapters 8 and 9 we consider next the parabolic system of inequalities

$$\begin{cases} 0 \leq u_t - \Delta u \leq v^p \\ 0 \leq v_t - \Delta v \leq u^q \end{cases} \quad \text{in } \Omega \times (0, 1),$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$. In order to carry out our study, new heat potential estimates are employed.

In order to provide the reader with all necessary tools for our approach, three Appendices are included, in which we present useful estimates for the heat kernel and nonlinear potential estimates for heat and Riesz potentials. To obtain these estimates, we include new Sobolev and Hedberg type inequalities for heat potentials.

The book is self-contained and most of the chapters can be read independently; only basic knowledge of functional analysis, measure theory, and Sobolev spaces is required. In order to offer a better perspective to the material developed in this monograph, each chapter is concluded with a section containing historical notes and related comments. The material in this book stems from the authors' original results in the study of singular phenomena arising in semilinear partial differential inequalities and from their long interest in this field. This monograph is intended for graduate and Ph.D. students interested in modern nonlinear analysis, researchers in pure and applied mathematics, and engineers involved in the field of PDEs.

We are grateful to Roger Astley of Cambridge University Press for his assistance in the preparation of this monograph.

Marius Ghergu
Steven D. Taliaferro
May 2015

Contents

1
Representation formulae for singular solutions of polyharmonic and parabolic inequalities

1.1 Introduction

This chapter is devoted to various integral representation formulae for singular solutions of polyharmonic and parabolic inequalities. This topic forms the building block in our study of isolated singularities for differential inequalities. Although integral representation formulae have been around for two centuries (a remarkable example is the famous Poisson integral formula for the Laplace's equation in the ball), in nonlinear PDEs they have been employed only in the last few decades. In this chapter, the reader will be gradually introduced to the topic of integral representation of distributional or classical solutions of linear differential inequalities. We start with a basic representation formula for superharmonic functions in a punctured ball. Then we extend such a result to solutions of $-\Delta^m u \geq 0$ in $B_1(0) \setminus \{0\} \subset \mathbb{R}^n$, $m, n \geq 1$. Finally, the integral representation for singular solutions corresponding to the heat operator is discussed. A common feature of all these results is that the integral operator in the representation formulae of the solution contains as a kernel the fundamental solution of the differential operator under consideration.

1.2 Harmonic inequalities in the punctured ball

In this section we consider C^2 nonnegative solutions of the harmonic inequality

$$-\Delta u \geq 0 \quad \text{in} \quad B_2(0) \setminus \{0\} \subset \mathbb{R}^n,\ n \geq 2. \tag{1.2.1}$$

According to the following theorem, these solutions satisfy representation formula (1.2.2) below.

Theorem 1.1 *Suppose u is a nonnegative C^2 solution of* (1.2.1) *and let $f = -\Delta u$. Then, $u, f \in L^1(B_1(0))$ and there exist a nonnegative constant m and a harmonic function $h : B_1(0) \to \mathbb{R}$ such that*

$$u(x) = m\Phi(|x|) + N(x) + h(x) \quad \textit{in} \quad B_1(0) \setminus \{0\}, \tag{1.2.2}$$

where

$$N(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n} \displaystyle\int_{|y|<1} |x-y|^{2-n} f(y)dy & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \displaystyle\int_{|y|<1} \log\left(\dfrac{2}{|x-y|}\right) f(y)dy & \text{if } n = 2, \end{cases} \tag{1.2.3}$$

and

$$\Phi(r) = \begin{cases} r^{2-n} & \text{if } n \geq 3, \\ \log \dfrac{1}{r} & \text{if } n = 2. \end{cases} \tag{1.2.4}$$

In (1.2.3), ω_n is the volume of the unit ball in $\mathbb{R}^n$.

For the proof of Theorem 1.1 we will need the following lemma.

Lemma 1.2 *Suppose v is a harmonic function in $B_1(0) \setminus \{0\}$ such that*

$$\int_{|x|<\varepsilon} |v(x)|dx = o(\varepsilon) \quad \text{as } \varepsilon \to 0^+. \tag{1.2.5}$$

Then $v - \beta\Phi(|x|)$ has a harmonic extension to $B_1(0)$ for some $\beta \in \mathbb{R}$.

Proof Let $\bar{v}(r)$ be the spherical average of v over the sphere $\partial B_r(0)$. Since v is harmonic in $B_1(0) \setminus \{0\}$, there exist $b, \beta \in \mathbb{R}$ such that

$$\bar{v}(r) = \beta\Phi(r) + b \quad \text{for all } 0 < r < 1. \tag{1.2.6}$$

Let $\eta \in C^\infty(\mathbb{R})$ be a decreasing function such that $\eta(t) = 1$ for $t \leq 1/2$ and $\eta(t) = 0$ for $t > 1$. For $\varepsilon > 0$ define $\psi_\varepsilon : \mathbb{R}^n \to [0, 1]$ by $\psi_\varepsilon(x) = \eta(|x|/\varepsilon)$.

Let now $\varphi \in C_0^\infty(B_1(0))$ and $\widehat{\varphi} = \varphi - \varphi(0)$. Then for small $\varepsilon > 0$ we have

$$\begin{aligned} \int (v - \bar{v}(|x|))\Delta\varphi &= \int (v - \bar{v}(|x|))[\Delta(\varphi\psi_\varepsilon) + \Delta(\varphi(1-\psi_\varepsilon))] \\ &= \int (v - \bar{v}(|x|))\Delta(\varphi\psi_\varepsilon) = I_1(\varepsilon) + \varphi(0)I_2(\varepsilon), \end{aligned} \tag{1.2.7}$$

where

$$I_1(\varepsilon) = \int (v - \bar{v})\Delta(\widehat{\varphi}\psi_\varepsilon) \quad \text{and} \quad I_2(\varepsilon) = \int (v - \bar{v})\Delta\psi_\varepsilon.$$

Since $\Delta\psi_\varepsilon$ is radial about the origin, $I_2(\varepsilon) = 0$ for small $\varepsilon > 0$. Also, since $\max_{|x|<\varepsilon} |\Delta(\widehat{\varphi}\psi_\varepsilon)| = O(\varepsilon^{-1})$ as $\varepsilon \to 0^+$, it follows from (1.2.5) and (1.2.6) that $I_1(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Hence letting $\varepsilon \to 0^+$ in (1.2.7) completes the proof. □

Proof of Theorem 1.1 Let $\bar{u}(r)$ denote the average of u over $\partial B_r(0)$. Averaging and integrating (1.2.1) we get

$$r^{n-1}\bar{u}'(r) = \bar{u}'(1) + \frac{1}{n\omega_n} \int_{r<|x|<1} f(x)dx \quad \text{for all } 0 < r < 1. \tag{1.2.8}$$

We claim $f \in L^1(B_1(0))$ for otherwise there exists $r_0 \in (0,1)$ such that the right-hand side in (1.2.8) is larger than 1 for $0 < r < r_0$ and hence

$$\bar{u}(r_0) - \bar{u}(r) \geq \begin{cases} \dfrac{1}{n-2}\left(r^{2-n} - r_0^{2-n}\right) & \text{if } n \geq 3, \\ \log \dfrac{r_0}{r} & \text{if } n = 2, \end{cases}$$

for $0 < r < r_0$. In particular, the above inequality implies $\bar{u}(r_0) - \bar{u}(r) \to \infty$ as $r \to 0^+$ which contradicts $u \geq 0$ in $B_1(0) \setminus \{0\}$. Hence $f \in L^1(B_1(0))$ and thus from (1.2.8) there exists a finite constant m such that

$$\frac{\bar{u}(r)}{\Phi(r)} = m + o(1) \quad \text{as } r \to 0^+. \tag{1.2.9}$$

Since $u \geq 0$, from (1.2.9) we derive $m \geq 0$ and as $\varepsilon \to 0^+$ we have

$$\int_{|x|<\varepsilon} u(x)dx = \begin{cases} O(\varepsilon^2) & \text{if } n \geq 3, \\ O\left(\varepsilon^2 \log \dfrac{1}{\varepsilon}\right) & \text{if } n = 2. \end{cases} \tag{1.2.10}$$

In particular, (1.2.10) implies $u \in L^1(B_1(0))$. Let next $N(x)$ be defined by (1.2.3) for all $x \in \mathbb{R}^n \setminus \{0\}$. Then $N \in C^1(\mathbb{R}^n \setminus \{0\}) \cap L^1(B_1(0))$ and

$$-\Delta N = f \quad \text{in } \mathcal{D}'(B_1(0)).$$

Using the identity

$$\frac{1}{|\partial B_r(0)|} \int_{|x|=r} \Phi(|x-y|)dS = \begin{cases} \Phi(|y|) & \text{if } |y| > r, \\ \Phi(r) & \text{if } |y| < r, \end{cases}$$

we easily find

$$\bar{N}(r) = o(\Phi(r)) \quad \text{as } r \to 0^+. \tag{1.2.11}$$

By (1.2.10) and (1.2.11) we now obtain

$$\int_{|x|<\varepsilon} |u(x) - N(x)|dx \leq \int_{|x|<\varepsilon} u(x)dx + \int_{|x|<\varepsilon} N(x)dx = \begin{cases} O(\varepsilon^2) & \text{if } n \geq 3, \\ O\left(\varepsilon^2 \log \dfrac{1}{\varepsilon}\right) & \text{if } n = 2, \end{cases}$$

as $\varepsilon \to 0^+$. It therefore follows from Lemma 1.2 that $u - N(x) - \beta\Phi(|x|)$ has a harmonic extension to $B_1(0)$. That is,

$$u = \beta\Phi(|x|) + N(x) + h(x) \quad \text{in } B_1(0) \setminus \{0\}, \tag{1.2.12}$$

where $h : B_1(0) \to \mathbb{R}$ is a harmonic function. Averaging (1.2.12) and using (1.2.11) and (1.2.9) we get $\beta = m \geq 0$ which completes our proof. □

The L^1-regularity of Δu is still true when dealing with distributional solutions that exhibit higher dimensional singularity set. This is illustrated by the following result.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\Sigma \subset\subset \Omega$ be a closed set of zero Newtonian capacity and assume that $u, f \in L^1_{loc}(\Omega \setminus \Sigma)$ are two nonnegative functions such that*

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma).$$

Then $u, f \in L^1_{loc}(\Omega)$ and

$$-\Delta u \geq f \quad \text{in } \mathcal{D}'(\Omega).$$

Proof For $j \geq 1$ let $u_j = \min\{u, j\}$ and $f_j = f\chi_{\{u<j\}}$. By Kato's inequality [69] we have

$$-\Delta u_j \geq f_j \quad \text{in } \mathcal{D}'(\Omega \setminus \Sigma). \tag{1.2.13}$$

Since $-\Delta u_j$ is a nonnegative distribution in $\Omega \setminus \Sigma$, it can be extended to a nonnegative measure on $\Omega \setminus \Sigma$. Also, the boundedness of u_j combined with a Gagliardo–Nirenberg type inequality yields $u_j \in H^1_{loc}(\Omega \setminus \Sigma)$. We claim that $u_j \in H^1_{loc}(\Omega)$. To see this, let $\phi \in C^\infty_c(\Omega)$ and let $\{\phi_k\} \subset C^\infty_c(\Omega \setminus \Sigma)$ be such that $\phi_k \to \phi$ in $H^1(\Omega)$. This is possible since $\text{cap}_\Omega(\Sigma) = 0$ (e.g., $\phi_k = \phi(1 - \chi_k)$ where $\chi_k = 1$ near Σ and $\|\chi_k\|_{H^1} \to 0$). We then have

$$\begin{aligned}
\int |\nabla u_j|^2 \phi_k^2 &\leq -e^j \int \phi_k^2 \nabla(e^{-u_j}) \cdot \nabla u_j \\
&= e^j \left(2 \int e^{-u_j} \phi_k \nabla \phi_k \cdot \nabla u_j + \int e^{-u_j} \Delta u_j \phi_k^2 \right) \\
&\leq 2e^{2j} \int e^{-u_j} |\nabla \phi_k|^2 + \frac{1}{2} \int e^{-u_j} |\nabla u_j|^2 \phi_k^2 \\
&\leq 2e^{2j} \int e^{-u_j} |\nabla \phi_k|^2 + \frac{1}{2} \int |\nabla u_j|^2 \phi_k^2
\end{aligned}$$

Hence

$$\int |\nabla (u_j \phi_k)|^2 \leq C_j \int |\nabla \phi_k|^2 \quad \text{for all } j, k \geq 1,$$

where $C_j > 0$. Passing to the limit with $k \to \infty$ in the above inequality we find $u_j \in H^1_{loc}(\Omega)$ for all $j \geq 1$. We next claim that

$$-\Delta u_j \geq f_j \quad \text{in } \mathcal{D}'(\Omega). \tag{1.2.14}$$

Indeed, let us notice first that from (1.2.13) we have

$$\int u_j(-\Delta \phi_k) \geq \int f_j \phi_k. \tag{1.2.15}$$

Using $u_j \in H^1_{loc}(\Omega)$ we deduce

$$\int u_j(-\Delta \phi_k) = \int \nabla u_j \cdot \nabla \phi_k \to \int \nabla u_j \cdot \nabla \phi = -\int u_j \Delta \phi \quad \text{as } k \to \infty.$$

Passing to the limit with $k \to \infty$ in (1.2.15) we obtain (1.2.14).