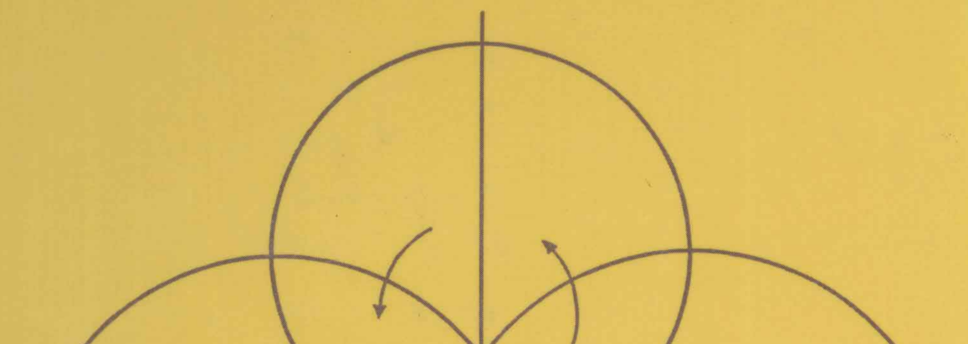
Toshitsune Miyake

Modular Forms



Springer-Verlag

Toshitsune Miyake

Modular Forms

With 11 Figures

Springer-Verlag Berlin Heidelberg New York
London Paris Tokyo Hong Kong

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Title of the original Japanese edition: Hokei keishiki to seisūron

Publisher: Kinokuniya Company Ltd., Tokyo
English translation rights arranged with Koji Doi and Toshitsune Miyake through Kinokuniya Company Ltd., Tokyo

Mathematics Subject Classification (1980): 10D12, 10D07, 10D24, 10D40

ISBN 3-540-50268-8 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-50268-8 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data.
Miyake, Toshitsune, 1944–
[Hokei keishiki to seisūron. English]
Modular forms / Toshitsune Miyake. p. cm.
Translation of: Hokei keishiki to seisūron. Includes bibliography and index.
ISBN 0-387-50268-8 (U.S.)
1. Forms, Modular I. Title
QA243.M6813 1989 512'.522–dc19 89-4121 CIP

Printed in the United States of America

Typesetting: Macmillan India Ltd., Bangalore
2141/3140-543210 Printed on acid-free paper

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Preface

Modular forms play an essential role in Number Theory. Furthermore the importance of modular forms has continued to grow in many areas of mathematics including the infinite dimensional representation theory of Lie groups and finite group theory. The aim of this book is to introduce some basic theory of modular forms of one variable.

Originally this book was written in Japanese under the title "Automorphic forms and Number Theory" by Koji Doi and myself and published by Kinokuniya, Tokyo, in 1976. When the English translation was planned, the first named author proposed that only the chapters written mainly by me be translated together with some additional material and published under my sole authorship.

In Chapters 1 and 2, the general theory of Fuchsian groups, automorphic forms and Hecke algebras is discussed. In Chapter 3, I summarize some basic results on Dirichlet series which are necessary in the succeeding chapters. In Chapter 4, the classical theories of modular groups and modular forms are studied. Here the usefulness of Hecke operators as well as the remarkable relation between modular forms and Dirichlet series obtained by Hecke and Weil have been emphasized. Chapter 5 briefly reviews quaternion algebras and their unit groups, which are also Fuchsian groups and which play a role similar to that of modular groups in their application to number theory. Chapter 6 is devoted to the trace formulae of Hecke operators by Eichler and Selberg. The formulae have been generalized by many people including H. Shimizu, H. Hijikata and H. Saito. A formula computable by them is also offered. In our Japanese edition, as an introduction to the automorphic forms of several variables, Chapter 7 deals with Eisenstein series of Hilbert modular groups and the application to values of zeta-functions (following Siegel). As a result of important series of recent work by Shimura on Eisenstein series, I decided to rewrite it to introduce some of his results on Eisenstein series restricting it to only the case of one variable.

I should like to express my deepest gratitude to Professor Goro Shimura, who constructed the essential part of the arithmetic theory of automorphic functions, for his valuable suggestions and encouragement.

The translation of Chapters 1 through 6 was prepared by my colleague Professor Yoshitaka Maeda. He also corrected mistakes in the original text, and gave me many appropriate suggestions. I express my deep and sincere thanks to him for his collaboration. I also express my hearty thanks to Professor Haruzo Hida

whose lectures at Hokkaido University during 1983–84 were very helpful for the preparation of the present volume, and to Professor Hiroshi Saito and Dr. Masaru Ueda who kindly read the manuscript very carefully as a whole or in part and made many valuable suggestions.

Sapporo, February 1989 *Toshitsune Miyake*

Notation and Terminology

1. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For a rational prime p, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. We also denote by $\mathbb{R}_+$, $\mathbb{R}_-$ and $\mathbb{C}^1$, the set of positive real numbers, the set of negative real numbers and the set of complex numbers with absolute value 1, respectively:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\},\ \mathbb{R}_- = \{x \in \mathbb{R} \mid x < 0\},\ \mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

2. For a complex number z, we denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, the real part and the imaginary part of z, respectively. When z is a non-zero complex number, we denote by $\arg(z)$ the argument of z, which we specify by $-\pi < \arg(z) \leqq \pi$. For a real number x, we denote by $[x]$ the largest integer not exceeding x. When x is a non-zero real number, $\mathrm{sgn}(x)$ denotes $+1$ or -1 according as $x > 0$ or $x < 0$.

3. For a ring R with unity 1, we denote by $R^\times$ the group of invertible elements in R. Further we write

$$M_n(R) = \text{the set of square matrices of degree } n \text{ over } R,$$

$$GL_n(R) = \{\alpha \in M_n(R) \mid \det(\alpha) \in R^\times\},$$

$$SL_n(R) = \{\alpha \in M_n(R) \mid \det(\alpha) = 1\}.$$

4. We denote by $\amalg$ the disjoint union of sets. For a finite set A, $|A|$ denotes the number of elements in A. We also denote by $\#\{\ldots\ldots\}$, the number of the elements of the set given by $\{\ldots\ldots\}$.

5. When $g_1, \ldots, g_m$ are elements of a group G, $\langle g_1, \ldots, g_m\rangle$ denotes the subgroup of G generated by $g_1, \ldots, g_m$. When $v_1, \ldots, v_m$ are vectors in a vector space V over a field K, $\langle v_1, \ldots, v_m\rangle$ denotes the subspace of V generated by $v_1, \ldots, v_m$. For mappings $g: A \to B$ and $f: B \to C$, we denote by $f \circ g$ the mapping of A to C given by

$$(f \circ g)(a) = f(g(a)) \qquad (a \in A).$$

Lecture Notes in Mathematics

Editors: A. Dold, B. Eckmann

Volume 1231

E.-U. Gekeler, Bonn, Germany

Drinfeld Modular Curves

1986. XIV, 107 pages. ISBN 3-540-17201-7

Contents: Notations. - Drinfeld Modules. - Lattices. - Partial Zeta Functions. - Drinfeld Modules of Rank 1. - Modular curves over C. - Expansions around Cusps. - Modular Forms and Functions. - Complements. - Index. - List of Symbols. - Bibliography.

Volume 1254

S. Gelbart, Rehovot; **I. Piatetski-Shapiro,** Ramat Aviv, Israel; **S. Rallis,** Columbus, OH, USA

Explicit Constructions of Automorphic L-Functions

1987. VI, 152 pages. ISBN 3-540-17848-1

Contents: L-Functions for the Classical Groups. - L-Functions for G GL (n): Basic Identities and the Euler Product Expansion. The Local Functional Equation. - General Index. - Index of Notation.

Volume 1304

F. Q. Gouvea, University of São Paulo, Brazil

Arithmetic of p-adic Modular Forms

1988. VIII, 121 pages. ISBN 3-540-18946-7

Contents: p-adic Modular Forms: Level Structures and Trivializations. p-adic Modular Forms with Growth Conditions. Generalized p-adic Modular Functions. - Hecke and U Operators: Hecke Operators. The Frobenius Operator. The U Operator. Appendix: Hida's Theory of the Ordinary Part. - Galois Representations: Duality Theorems. Families of Modular Forms. Changing the Level. Deformations of Residual Eigenforms. Deformations of Galois Representations. The Modular Deformation Space. Further Questions.

Springer-Verlag Berlin
Heidelberg New York London
Paris Tokyo Hong Kong

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J. Neukirch, University of Regensburg, Germany

Class Field Theory

1986. VIII, 140 pages. (Grundlehren der mathematischen Wissenschaften, Band 280).
ISBN 3-540-15251-2

Contents: Group and Field Theoretic Foundations. - General Class Field Theory. - Local Class Field Theory. - Global Class Field Theory. - Zeta Functions and L-Series. - Literature. - Index.

Class field theory is the culmination of the development of number theory beginning with Gauß's reciprocity law and covering the work of Kummer, Dirichlet, Kronecker, Hilbert, Artin-Hasse. It contains the main achievements and the most significant results of the theory of algebraic number fields.

This Grundlehren volume offers a completely new approach to this theory requiring neither cohomological nor analytic methods, and provides a basic and direct way of understanding and proving the central theorem, namely the reciprocity law.

This exposition opens the door to other disciplines, such as the theory of formal groups and cohomology theory of number fields, and is fundamental for modern research in number theory.

Besides researchers, the book will be essential reading for graduate students with a background in Galois-Theory and Algebraic Number Theory.

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Contents

Chapter 1. The Upper Half Plane and Fuchsian Groups

We explain basic properties of the upper half plane **H** in § 1.1 through § 1.4. We introduce Fuchsian groups in § 1.5 which play an essential role throughout the book. In § 1.6 through § 1.8, we study the quotient spaces of **H** by Fuchsian groups and induce the structure of Riemann surfaces on them.

§ 1.1. The Group of Automorphisms of the Upper Half Plane

We denote by $\mathbb{P}$ the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and define the action of an element $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $GL_2(\mathbb{C})$ on $\mathbb{P}$ by

(1.1.1) $$\alpha z = \frac{az+b}{cz+d} \quad (z \in \mathbb{P}).$$

This mapping "$z \mapsto \alpha z$" is complex analytic from $\mathbb{P}$ into itself. We put

(1.1.2) $$j(\alpha, z) = cz + d \quad (z \in \mathbb{C}).$$

If $z \in \mathbb{C}$ and $j(\alpha, z) \neq 0$, then we have

(1.1.3) $$\alpha \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = j(\alpha, z) \begin{bmatrix} \alpha z \\ 1 \end{bmatrix}.$$

This equality also holds when considering each constituent as a meromorphic function. Calculating $\alpha\beta \begin{bmatrix} z \\ 1 \end{bmatrix}$ $(\alpha, \beta \in GL_2(\mathbb{C}))$ in two ways, we see that

$$j(\alpha, \beta z)\, j(\beta, z) \begin{bmatrix} \alpha(\beta z) \\ 1 \end{bmatrix} = j(\alpha\beta, z) \begin{bmatrix} (\alpha\beta) z \\ 1 \end{bmatrix}.$$

From this equality, we obtain

(1.1.4) $$(\alpha\beta) z = \alpha(\beta z) \qquad (\alpha, \beta \in GL_2(\mathbb{C}),\ z \in \mathbb{P}),$$

and

(1.1.5) $$j(\alpha\beta, z) = j(\alpha, \beta z)\, j(\beta, z) \qquad (\alpha, \beta \in GL_2(\mathbb{C}),\ z \in \mathbb{C}).$$

By (1.1.4), the mapping "$z \mapsto \alpha^{-1}z$" is the inverse mapping of "$z \mapsto \alpha z$", and therefore, "$z \mapsto \alpha z$" is an automorphism of the Riemann sphere $\mathbb{P}$. This automorphism is called a *linear fractional transformation.* Putting $\beta = \alpha^{-1}$ in (1.1.5), we see

$$j(\alpha^{-1}, z) = j(\alpha, \alpha^{-1}z)^{-1}. \tag{1.1.6}$$

Lemma 1.1.1. *A linear fractional transformation maps circles and lines on $\mathbb{C}$ into circles or lines on $\mathbb{C}$.*

Proof. We put for an element $\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $GL_2(\mathbb{C})$

$$C_\beta = \{z \in \mathbb{P} \mid |\beta z| = 1\}.$$

Since z belongs to $C_\beta \cap \mathbb{C}$ if and only if $|az+b| = |cz+d|$, $C_\beta \cap \mathbb{C}$ is a line (if $a = c$) or a circle (the Appolonius circle). Conversely it is easily seen that circles and lines on $\mathbb{C}$ can be expressed as $C_\beta \cap \mathbb{C}$ with some $\beta \in GL_2(\mathbb{C})$. Let α be an element of $GL_2(\mathbb{C})$ and denote by $\alpha(C_\beta)$ the image of C_β by α. Since $\alpha(C_\beta) = C_{\beta\alpha^{-1}}$, $\alpha(C_\beta) \cap \mathbb{C}$ is again a circle or a line on $\mathbb{C}$. □

We define two domains $\mathbf{H}$ and $\mathbf{K}$ of $\mathbb{C}$ by

$$\mathbf{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

and

$$\mathbf{K} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

The domains $\mathbf{H}$ and $\mathbf{K}$ are called the *upper half plane* and the *unit disk*, respectively.

Lemma 1.1.2. *The upper half plane* $\mathbf{H}$ *and the unit disk* $\mathbf{K}$ *are complex analytically isomorphic.*

Proof. Put $\rho = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$. Then "$z \mapsto \rho z$" is an automorphism of $\mathbb{P}$, and satisfies

$$|\rho z| = \left|\frac{z-i}{z+i}\right| < 1 \qquad (z \in \mathbf{H}).$$

Since we see

$$\operatorname{Im}(\rho^{-1}w) = \operatorname{Im}\left(i\frac{w+1}{-w+1}\right) = \frac{1-|w|^2}{|1-w|^2} > 0 \qquad (w \in \mathbf{K}),$$

ρ gives an analytic isomorphism of $\mathbf{H}$ onto $\mathbf{K}$. □

We are interested in functions on $\mathbf{H}$ which satisfy certain transformation equations for automorphisms of $\mathbf{H}$. (We say that *they have automorphy.*) We first study automorphisms of $\mathbf{H}$. We denote by $\operatorname{Aut}(\mathbf{H})$ and $\operatorname{Aut}(\mathbf{K})$ the groups of all (complex analytic) automorphisms of $\mathbf{H}$ and $\mathbf{K}$, respectively. If $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$, and $z \in \mathbf{H}$, then

$$\mathrm{Im}(\alpha z) = \frac{\det(\alpha)\,\mathrm{Im}(z)}{|cz+d|^2}. \tag{1.1.7}$$

In particular, if $\det(\alpha)>0$, then we have $\mathrm{Im}(\alpha z)>0$, and therefore, "$z \mapsto \alpha z$" induces an automorphism of $\mathbf{H}$. We put

$$GL_2^+(\mathbb{R}) = \{\alpha \in GL_2(\mathbb{R}) \mid \det(\alpha)>0\},$$

and denote by $\iota(\alpha)$ $(\alpha \in GL_2^+(\mathbb{R}))$ the automorphism "$z \mapsto \alpha z$" of $\mathbf{H}$. Then it follows from (1.1.4) that this mapping

$$\iota\colon GL_2^+(\mathbb{R}) \ni \alpha \mapsto \iota(\alpha) \in \mathrm{Aut}(\mathbf{H})$$

is a group-homomorphism.

Now we put

$$SO_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \middle| \, 0 \leqq \theta < 2\pi \right\}.$$

We identify $a \in \mathbb{R}^\times$ with $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in GL_2^+(\mathbb{R})$. If for $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$, $\iota(\alpha)$ is the identity of $\mathbf{H}$, then α belongs to $\mathbb{R}^\times$, since $cz^2+(d-a)z-b=0$ for any $z \in \mathbf{H}$. Now we have

Theorem 1.1.3. (1) *For any $z \in \mathbf{H}$, there exists an element α in $SL_2(\mathbb{R})$ satisfying $\alpha i = z$.*

(2) *The homomorphism ι induces an isomorphism*

$$GL_2^+(\mathbb{R})/\mathbb{R}^\times \simeq SL_2(\mathbb{R})/\{\pm 1\} \simeq \mathrm{Aut}(\mathbf{H}).$$

(3) $SO_2(\mathbb{R}) = \{\alpha \in SL_2(\mathbb{R}) \mid \alpha i = i\}$

and

$$\mathbb{R}^\times \cdot SO_2(\mathbb{R}) = \{\alpha \in GL_2^+(\mathbb{R}) \mid \alpha i = i\}.$$

Proof. For any $z=x+yi \in \mathbf{H}$, put $\alpha = \sqrt{y}^{-1}\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$. Then α belongs to $SL_2(\mathbb{R})$ and $\alpha i = z$; this proves (1). The first isomorphism of (2) is obvious. To see the second isomorphism, we have only to verify the surjectivity. For this purpose, it is sufficient to show that if an element ψ of $\mathrm{Aut}(\mathbf{H})$ satisfies $\psi(i)=i$, then there exists an element β in $SO_2(\mathbb{R})$ such that $\psi = \iota(\beta)$. In fact, for each element $\phi \in \mathrm{Aut}(\mathbf{H})$, we get an element $\alpha \in SL_2(\mathbb{R})$ satisfying $\alpha^{-1}\phi(i)=i$ by (1). Then taking $\iota(\alpha^{-1})\phi$ in place of ψ, we have $\phi = \iota(\alpha\beta)$ for some $\beta \in SO_2(\mathbb{R})$; this implies ι is surjective. Now let ψ be an element of $\mathrm{Aut}(\mathbf{H})$ such that $\psi(i)=i$. We put

$$\rho(z) = (z-i)/(z+i) \qquad (z \in \mathbf{H}),$$

which is an isomorphism of $\mathbf{H}$ onto $\mathbf{K}$. Since $\rho(i)=0$, $\eta = \rho\psi\rho^{-1}$ is an automorphism of $\mathbf{K}$ such that $\eta(0)=0$. Applying Schwarz's theorem to η and η^{-1}, we see that

$$|\eta(w)| = |w| \qquad (w \in \mathbf{K}).$$

A further application of Schwarz's theorem shows that there exists $\theta (0 \leqq \theta < \pi)$ such that

$$\eta(w) = e^{2\theta i} w \qquad (w \in \mathbf{K}).$$

Thus pulling back the function η by ρ^{-1} to $\mathbf{H}$, we see that

$$\psi(z) = \rho^{-1} \eta \rho(z) = \frac{(\cos \theta)z + \sin \theta}{(-\sin \theta)z + \cos \theta} \qquad (z \in \mathbf{H});$$

namely $\psi = \iota(k_\theta)$ with $k_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \in SO_2(\mathbb{R})$. This implies (2) and (3). □

Now let us consider $\mathrm{Aut}(\mathbf{K})$. Put

$$SU(1,1) = \left\{ g \in SL_2(\mathbb{C}) \,\middle|\, {}^t\bar{g} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$
$$= \left\{ g = \begin{bmatrix} u & v \\ \bar{v} & \bar{u} \end{bmatrix} \,\middle|\, u, v \in \mathbb{C}, \quad |u|^2 - |v|^2 = 1 \right\}.$$

Since

(1.1.8) $$\rho SL_2(\mathbb{R}) \rho^{-1} = SU(1,1), \quad \rho = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},$$

we see by Lemma 1.1.2 and Theorem 1.1.3(2)

(1.1.9) $$\mathrm{Aut}(\mathbf{K}) \simeq SU(1,1)/\{\pm 1\}.$$

§ 1.2. Actions of Groups

In this section, we prepare general theory on topological spaces and transformation groups to apply it to the upper half plane $\mathbf{H}$.

Let G be a group and X a topological space (resp. a complex domain). We say that G *acts on* X if there exists a mapping

$$G \times X \ni (g, x) \mapsto gx \in X$$

satisfying the following three conditions:

(i) *for each element g of G, "$X \ni x \mapsto gx \in X$" is a continuous (resp. complex analytic) mapping;*

(ii) $(gh)x = g(hx)$ *for two elements g and h of G;*

(iii) *for the unit element* 1 *of G, $1x = x$ for any element x of X.*

Since for any element g of G, "$x \mapsto g^{-1}x$" is the inverse mapping of "$x \mapsto gx$", we see that if G acts on X, then

(i′) *for each element g of G, "$X \ni x \mapsto gx \in X$" is a topological (resp. complex analytic) automorphism of X.*

We assume hereafter that G acts on X. For an element x of X we put

$$G_x = \{g \in G \mid gx = x\},$$

and call it the *stabilizer of* x. It is obvious that for any element g of G,

$$G_{gx} = gG_x g^{-1}. \tag{1.2.1}$$

An element x of X is called a *fixed point of* $g \in G$ if

$$gx = x.$$

This is equivalent to saying $g \in G_x$. Further for each element x of X, a subset of X defined by

$$Gx = \{gx \mid g \in G\}$$

is called the *G-orbit of* x. The set of all G-orbits in X is denoted by $G\backslash X$. Since either $Gx = Gy$ or $Gx \cap Gy = \varnothing$ for any two elements x, y of X, X can be expressed as a disjoint union of G-orbits:

$$X = \coprod Gx.$$

In particular, if X itself is a G-orbit, then we say that *G acts transitively on X*. This is equivalent to saying that

(1.2.2) *for any two elements x, y of X, there exists an element g of G such that $gx = y$.*

Therefore, it follows from (1.2.1) that if G acts transitively on X, then all the stabilizers are conjugate.

Now let us denote by π the canonical mapping of X onto the set of all G-orbits $G\backslash X$; in other words, π is a mapping which corresponds any element x of X to the element Gx of $G\backslash X$:

$$X \ni x \mapsto \pi(x) = Gx \in G\backslash X.$$

We induce the strongest topology on $G\backslash X$ under which the above projection π is continuous. More precisely, it is given by defining that a subset U of $G\backslash X$ is open if and only if the inverse image $\pi^{-1}(U)$ of U by π is open in X. The topological space $G\backslash X$ with this topology is called the *quotient space of X by G*. Since for an open subset U of X, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU \qquad gU = \{gu \mid u \in U\},$$

$\pi(U)$ is again open in $G\backslash X$. Thus π is an open continuous mapping of X onto the quotient space $G\backslash X$.

If a group G has a topological structure of a Hausdorff space, and the two mappings

$$G \times G \ni (g, h) \mapsto gh \in G, \qquad G \ni g \mapsto g^{-1} \in G$$

are continuous with respect to its topology, then G is called a *topological group*. Let

G be a topological group and assume that G acts on X. Then we say that a topological group G acts on a topological space X, if the additional condition (iv) below is satisfied:

(iv) $G \times X \ni (g, x) \mapsto gx \in X$ *is continuous.*

If a topological group G acts on a topological space X, then all the stabilizers are closed subgroups of G. Conversely, let G be a topological group and K a closed subgroup of G. Then K acts on G by right multiplication. We denote by G/K the quotient space of G by K, and call it the *space of the right cosets of G by K.*

Theorem 1.2.1. *Assume that a topological group G acts transitively on a topological space X. If G is a locally compact group with a countable basis, and X is a locally compact Hausdorff space, then for each element $x \in X$, the space of the right cosets G/G_x is homeomorphic to X by the correspondence "$gG_x \mapsto gx$".*

Proof. It is obvious that the correspondence is bijective. Thus it is sufficient to show that it is bicontinuous. From the definition of the topology on G/G_x, it is equivalent to saying that "$g \mapsto gx$" is an open continuous mapping of G to X. The continuity is obvious by definition, and therefore it is sufficient to show that this mapping is also open. Let us prove that for any open set U of G, $Ux = \{gx \mid g \in U\}$ is also open in X. Let gx ($g \in U$) be any point of Ux. Take a compact neighborhood V of the unit element of G so that $V^{-1} = V$ and $gV^2 \subset U$. Since G has a countable basis, there exist countably many elements g_n ($n = 1, 2, \ldots$) satisfying $G = \bigcup_{n=1}^{\infty} g_n V$. Put $W_n = g_n Vx$, then $X = \bigcup_{n=1}^{\infty} W_n$. Since W_n is a compact set in the Hausdorff space X, it is closed. Now suppose that no W_n contains an open subset. Since X is regular, we find inductively non-empty open subsets U_n so that the closures $\bar{U}_n$ are compact and

$$U_{n-1} - W_{n-1} \supset \bar{U}_n \qquad (n \geqq 2).$$

Then we see that $\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \ldots$. Since $\bigcap_{n=1}^{\infty} \bar{U}_n \neq \emptyset$ and $\bigcap_{n=1}^{\infty} \bar{U}_n$ has no common point with any W_n, this contradicts the fact $X = \bigcup_{n=1}^{\infty} W_n$. Hence there exists a set $W_m = g_m Vx$ which contains an open subset of X. Since $g_m Vx$ is homeomorphic to Vx, Vx also contains an open subset S. For an element h of V such that $hx \in S$, we have

$$gx \in gh^{-1}S \subset gV^2x \subset Ux.$$

Therefore gx is an interior point of Ux. This proves that Ux is open. □

Now Theorem 1.1.3 implies that the topological group $SL_2(\mathbb{R})$ acts transitively on the complex domain $\mathbf{H}$ and the stabilizer of i is $SO_2(\mathbb{R})$. Thus, applying the above theorem to $X = \mathbf{H}$ and $G = SL_2(\mathbb{R})$, we obtain the following

Corollary 1.2.2. *The space of the cosets $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ is homeomorphic to* **H** *by the correspondence "$\alpha SO_2(\mathbb{R}) \mapsto \alpha i$".*