PLANE AND SPHERICAL TRIGONOMETRY

LYMAN AND GODDARD

WITH TABLES

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BY

ELMER A. LYMAN
MICHIGAN STATE NORMAL COLLEGE

AND

EDWIN C. GODDARD
UNIVERSITY OF MICHIGAN

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PREFACE.

Many American text-books on trigonometry treat the solution of triangles quite fully; English text-books elaborate analytical trigonometry; but no book available seems to meet both needs adequately. To do that is the first aim of the present work, in the preparation of which nearly everything has been worked out and tested by the authors in their classes.

The work entered upon, other features demanded attention. For some unaccountable reason nearly all books, in proving the formulæ for functions of $\alpha \pm \beta$, treat the same line as both positive and negative, thus vitiating the proof; and proofs given for acute angles are (without further discussion) supposed to apply to all angles, or it is suggested that the student can draw other figures and show that the formulæ hold in all cases. As a matter of fact the average student cannot show anything of the kind; and if he could, the proof would still apply only to combinations of conditions the same as those in the figures actually drawn. These difficulties are avoided by so wording the proofs that the language applies to figures involving any angles, and to avoid drawing the indefinite number of figures necessary fully to establish the formulæ geometrically, the general case is proved algebraically (see page 58).

Inverse functions are introduced early, and used constantly. Wherever computations are introduced they are made by means of logarithms. The average student, using logarithms for a short time and only at the end of the subject, straightway forgets what manner of things they are. It is hoped, by dint of much practice, extended over as long a time as possible, to give the student a command of logarithms that will stay. The fundamental formulæ of trigonometry must be memorized. There is no substitute for this. For this purpose oral work is introduced, and there are frequent lists of review problems involving all principles and formulæ previously developed. These lists serve the further purpose of throwing the student on his own resources, and compelling him to find in the problem itself, and not in any model solution, the key to its solution, thus developing power, instead of ability to imitate. To the same end, in the solution of triangles, divisions and subdivisions into cases are abandoned, and the student is thrown on his own judgment to determine which of the three possible sets of formulæ will lead to the solutions with the data given. Long experience justifies this as clearer and simpler. The use of checks is insisted upon in all computations.

For the usual course in plane trigonometry Chapters I–VII, omitting Arts. 26, 27, contain enough. Articles marked * (as Art. * 26) may be omitted unless the teacher finds time for them without neglecting the rest of the work. Classes that can accomplish more will find a most interesting field opened in the other chapters. More problems are provided than any student is expected to solve, in order that different selections may be assigned to different students, or to classes in different years. *Do not assign work too fast. Make sure the student has memorized and can use each preceding formula, before taking up new ones.*

No complete acknowledgment of help received could here be made. The authors are under obligation to many for general hints, and to several who, after going over the proof with care, have given valuable suggestions. The standard works of Levett and Davison, Hobson, Henrici and Treutlein, and others have been freely consulted, and while many of the problems have been prepared by the authors in their class-room work, they have not hesitated to take, from such standard collections as writers generally have drawn upon, any problems that seemed better adapted than others to the work. Quality has not been knowingly sacrificed to originality. Corrections and suggestions will be gladly received at any time.

E. A. L., YPSILANTI.
E. C. G., ANN ARBOR.

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PLANE TRIGONOMETRY.

CHAPTER I.

ANGLES — MEASUREMENT OF ANGLES.

1. Angles. It is difficult, if not impossible, to define an angle. This difficulty may be avoided by telling how it is formed. *If a line revolve about one of its points, an angle is generated*, the magnitude of the angle depending on the amount of the rotation.

Thus, if one side of the angle θ, as OR, be originally in the position OX, and be revolved about the point O to the position in the figure, the angle XOR is generated. OX is called the *initial* line, and any position of OR the *terminal* line of the angle formed. The angle θ is considered *positive if generated by a counter-clockwise rotation of* OR, and hence *negative if generated by a clockwise rotation.* The magnitude of θ depends on the amount of rotation of OR, and since the amount of such rotation may be unlimited, there is no limit to the possible magnitude of angles, for, evidently, the revolving line may reach the position OR by rotation through an acute angle θ, and, likewise, by rotation through once, twice, $\cdots$, n times 360°, plus the acute angle θ. So that XOR may mean the acute angle θ, $\theta + 360°$, $\theta + 720°$, $\cdots$, $\theta + n \cdot 360°$.

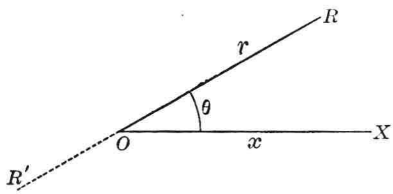

FIG. 1.

In reading an angle, read first the initial line, then the terminal line. Thus in the figure the acute angle XOR, or xr, is a positive angle, and ROX, or rx, an equal negative angle.

Ex. **1**. Show that if the initial lines for $\frac{1}{2}$, $\frac{9}{2}$, $\frac{25}{2}$, $-\frac{7}{2}$, right angles are the same, the terminal lines may coincide.

2. Name four other angles having the same initial and terminal lines as $\frac{1}{3}$ of a right angle; as $\frac{5}{4}$ of a right angle; as $\frac{3}{2}$ of a right angle.

2. Rectangular axes. Any plane surface may be divided by two perpendicular straight lines XX' and YY' into four portions, or *quadrants*.

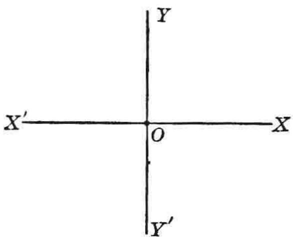

FIG. 2.

XX' is known as the *x-axis*, YY' as the *y-axis*, and the two together are called *axes of reference*. Their intersection O is the *origin*, and the four portions of the plane surface, XOY, YOX', $X'OY'$, $Y'OX$, are called respectively the *first*, *second*, *third*, and *fourth quadrants*. The position of any point in the plane is determined when we know its *distances* and *directions* from the axes.

3. Any direction may be considered positive. Then the opposite direction must be negative. Thus, if AB represents any positive line, BA is an equal negative line. Mathematicians usually consider *lines measured in the same direction as OX or OY* (Fig. 2) *as positive. Then lines measured in the same direction as OX' or OY' must be negative.*

The distance of any point from the y-axis is called the *abscissa*, its distance from the x-axis the *ordinate*, of that point; the two together are the *coördinates* of the point, usually denoted by the letters x and y respectively, and written (x, y).

When taken with their proper signs, the coördinates define completely the position of the point. Thus, if the point P is $+a$ units from YY', and $+b$ units from XX', any convenient unit of length being chosen, the position of P is known. For we have only to measure a distance ON equal to a units along OX, and then from N measure a distance b units parallel to OY, and we arrive at the position of the point P, (a, b). In like manner we may locate P', $(-a, b)$, in the second quadrant, P'', $(-a, -b)$, in the third quadrant, and P''', $(a, -b)$, in the fourth quadrant.

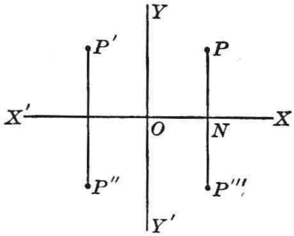

FIG. 3.

Ex. Locate $(2, -2)$; $(0, 0)$; $(-8, -7)$; $(0, 5)$; $(-2, 0)$; $(2, 2)$; (m, n).

4. If OX is the initial line, θ is said to be an *angle of the first*, *second*, *third*, or *fourth quadrant*, according as its terminal line is in the first, second, third, or fourth quadrant. It is clear that as OR rotates its *quality* is in no way affected, and hence it is *in all positions considered positive*, and its extension through O, OR', negative.

The student should notice that the initial line may take any position and revolve in either direction. While it is customary to consider the counter-clockwise rotation as forming a positive angle, yet the conditions of a figure may be such that a positive angle may be generated by a clockwise rotation. Thus the angle XOR in each figure may be traced as a positive angle by revolving the initial line OX to the position OR. No confusion can result if the fact is clear that when an angle is read XOR, OX is considered a positive line revolving to the position OR. OX' and OR' then are negative lines in whatever directions drawn. These conceptions are mere matters of agreement, and the agreement may be determined in a particular case by the conditions of the problem quite as well as by such general agreements of mathematicians as those referred to in Arts. 3 and 4 above.

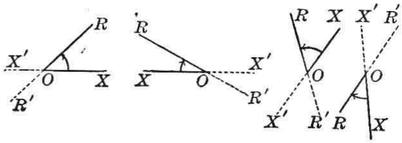

FIG. 4.

5. Measurement. All measurements are made in terms of some fixed standard adopted as a unit. This unit must be of the same kind as the quantity measured. Thus, length is measured in terms of a unit length, surface in terms of a unit surface, weight in terms of a unit weight, value in terms of a unit value, an angle in terms of a unit angle.

The measure of a given quantity is the number of times it contains the unit selected.

Thus the area of a given surface in square feet is the number of times it contains the unit surface 1 sq. ft.; the length of a road in miles, the number of times it contains the unit length 1 mi.; the weight of a cargo of iron ore in tons, the number of times it contains the unit weight 1 ton; the value of an estate, the number of times it contains the unit value $1.

The same quantity may have different measures, according to the unit chosen. So the measure of 80 acres, when the unit surface is 1 acre, is 80, when the unit surface is 1 sq. rd., is 12,800, when the unit surface is 1 sq. yd., is 387,200. What is its measure in square feet?

6. The essentials of a good unit of measure are:

1. *That it be invariable, i.e.* under all conditions bearing the same ratio to equal magnitudes.

2. *That it be convenient* for practical or theoretical purposes.

3. *That it be of the same kind as the quantity measured.*

7. Two systems of measuring angles are in use, the *sexagesimal* and the *circular*.

The *sexagesimal* system is used in most practical applications. The right angle, the unit of measure in geometry, though it is invariable, as a measure is too large for convenience. Accordingly it is divided into 90 equal parts, called *degrees*. The degree is divided into 60 *minutes*, and the minute into 60 *seconds*. Degrees, minutes, seconds, are indicated by the marks $^\circ$ $'$ $''$, as $36^\circ\ 20'\ 15''$.

The division of a right angle into hundredths, with subdivisions into hundredths, would be more convenient. The French have proposed such a *centesimal* system, dividing the right angle into 100 grades, the grade into 100 minutes, and the minute into 100 seconds, marked $^{g}\ '\ ''$, as $50^{g}\ 70'\ 28''$. The great labor involved in changing mathematical tables, instruments, and records of observation to the new system has prevented its adoption.

8. The *circular* system is important in theoretical considerations. It is based on the fact that for a given angle the ratio of the length of its arc to the length of the radius of that arc is constant, *i.e.* for a fixed angle the ratio *arc : radius* is the same no matter what the length of the radius. In the figure, for the angle θ,

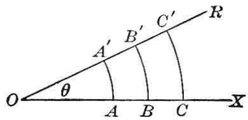

FIG. 5.

$$\frac{OA}{AA'}=\frac{OB}{BB'}=\frac{OC}{CC'}=\cdots.$$

That this ratio of arc to radius for a fixed angle is constant follows from the established geometrical principles:

1. The circumference of any circle is 2π times its radius.
2. Angles at the centre are in the same ratio as their arcs.

The Radian. It follows that an angle whose arc is equal in length to the radius is a constant angle for all circles, since in four right angles, or the perigon, there are always 2π such angles. *This constant angle, whose arc is equal in length to the radius, is taken as the unit angle of circular measure, and is called the radian.* From the definition we have

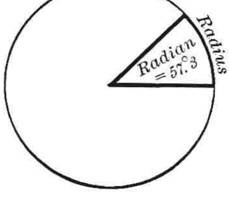

FIG. 6.

$$4 \text{ right angles} = 360° = 2\pi \text{ radians},$$

$$2 \text{ right angles} = 180° = \pi \text{ radians},$$

$$1 \text{ right angle} = 90° = \frac{\pi}{2} \text{ radians}.$$

π is a numerical quantity, 3.14159+, and not an angle. When we speak of 180° as π, 90° as $\frac{\pi}{2}$, etc., we always mean π *radians*, $\frac{\pi}{2}$ *radians*, etc.

9. To change from one system of measurement to the other we use the relation,

$$2\pi \text{ radians} = 360°.$$

$$\therefore\ 1 \text{ radian} = \frac{180°}{\pi} = 57°.2958-;$$

i.e. the radian is 57°.3, *approximately.*

Ex. **1.** Express in radians 75° 30′.

$$75°\ 30' = 75°.5;\ 1 \text{ radian} = 57°.3.$$

$$\therefore\ 75°\ 30' = \frac{75.5}{57.3} = 1.317 \text{ radians.}$$

2. Express in degree measure 3.6 radians.

$$1 \text{ radian} = 57°.3.$$

$$\therefore\ 3.6 \text{ radians} = 3.6 \times 57°.3 = 206°\ 16'\ 48''.$$

EXAMPLES.

1. Construct, approximately, the following angles: 50°, – 20°, 90°, 179°, – 135°, 400°, – 380°, 1140°, $\frac{\pi}{4}$ radians, $\frac{\pi}{3}$ radians, $-\frac{\pi}{6}$ radians, 3π radians, $-\frac{3\pi}{4}$ radians, $\frac{12\pi}{5}$ radians. Of which quadrant is each angle?

2. What is the measure of:

(*a*) $\frac{5}{4}$ of a right angle, when 30° is the unit of measure?
(*b*) an acre, when a square whose side is 10 rds. is the unit?
(*c*) m miles, when y yards is the unit?

3. What is the unit of measure, when the measure of $2\frac{1}{2}$ miles is 50?

4. The Michigan Central R.R. is 535 miles long, and the Ann Arbor R.R. is 292 miles long. Express the length of the first in terms of the second as a unit.

5. What will be the measure of the radian when the right angle is taken for the unit? Of the right angle when the radian is the unit?

6. In which quadrant is 45°? 10°? – 60°? 145°? 1145°? – 725°? Express each in right angles; in radians.

7. Express in sexagesimal measure

$$\frac{\pi}{3},\ \frac{\pi}{12},\ 1,\ 6.28,\ \frac{1}{\pi},\ \frac{7\pi}{3},\ -\frac{4\pi}{3}, \text{ radians.}$$

8. Express in each system an interior angle of a regular hexagon; an exterior angle.

9. Find the distance in miles between two places on the earth's equator which are $11° 15'$ apart. (The earth's radius is about 3963 miles.)

10. Find the length of an arc which subtends an angle of 4 radians at the centre of a circle of radius 12 ft. 3 in.

11. An arc 15 yds. long contains 3 radians. Find the radius of the circle.

12. Show that the hour and minute hands of a watch turn through angles of $30'$ and $6°$ respectively per minute; also find in degrees and in radians the angle turned through by the minute hand in 3 hrs. 20 mins.

13. Find the number of seconds in an arc of 1 mile on the equator; also the length in miles of an arc of $1'$ (1 knot).

14. Find to three decimal places the radius of a circle in which the arc of $71° 36' 3''.6$ is 15 in. long.

15. Find the ratio of $\frac{\pi}{6}$ to $5°$.

16. What is the shortest distance measured on the earth's surface from the equator to Ann Arbor, latitude $+42° 16' 48''$?

17. The difference of two angles is $10°$, and the circular measure of their sum is 2. Find the circular measure of each angle.

18. A water wheel of radius 6 ft. makes 30 revolutions per minute. Find the number of miles per hour travelled by a point on the rim.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

10. Trigonometry, as the word indicates, was originally concerned with the measurement of triangles. It now includes the analytical treatment of certain *functions of angles*, as well as the solution of triangles by means of certain relations between the functions of the angles of those triangles.

11. Function. If one quantity depends upon another for its value, the first is called a *function* of the second. It always follows that the second quantity is also a function of the first; and, in general, functions are so related that if one is constant the other is constant, and if either varies in value, the other varies. This relation may be extended to any number of mutually dependent quantities.

Illustration. If a train moves at a rate of 30 miles per hour, the distance travelled is a function of the rate and time, the time is a function of the rate and distance, and the rate is a function of the time and distance.

Again, the circumference of a circle is a function of the radius, and the radius of the circumference, for so long as either is constant the other is constant, and if either changes in value, the other changes, since circumference and radius are connected by the relation $C = 2\pi R$.

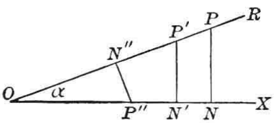

FIG. 7.

Once more, in the right triangle NOP, the ratio of any two sides is a function of the angle α, because all the right triangles of which α is one angle are similar, *i.e.* the ratio of two corresponding sides is constant so long as α is constant, and varies if α varies.

Thus, the ratios

$$\frac{NP}{OP}=\frac{N'P'}{OP'}=\frac{N''P''}{OP''}$$

and
$$\frac{ON}{NP}=\frac{ON'}{N'P'}=\frac{ON''}{N''P''}, \text{ etc.,}$$

depend on α for their values, *i.e.* are functions of α.

12. The trigonometric functions. In trigonometry six *functions of angles* are usually employed, called the *trigonometric functions.*

By definition *these functions are the six ratios between the sides of the triangle of reference of the given angle. The triangle of reference is formed by drawing from some point in the initial line, or the initial line produced, a perpendicular to that line meeting the terminal line of the angle.*

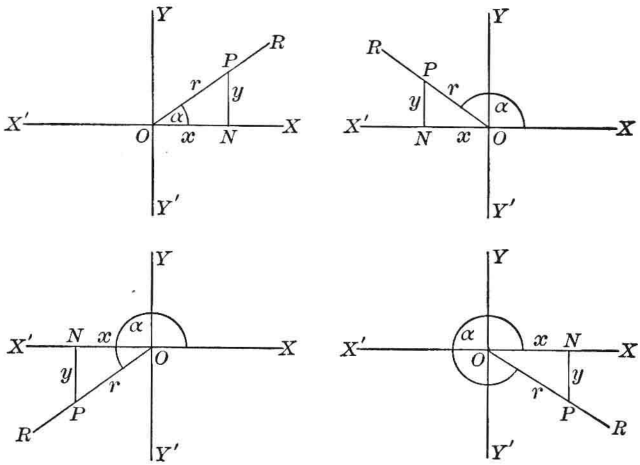

FIG. 8.

Let α be an angle of any quadrant. Each triangle of reference of α, NOP, is formed by drawing a perpendicular to OX, or OX produced, meeting the terminal line OR in P.