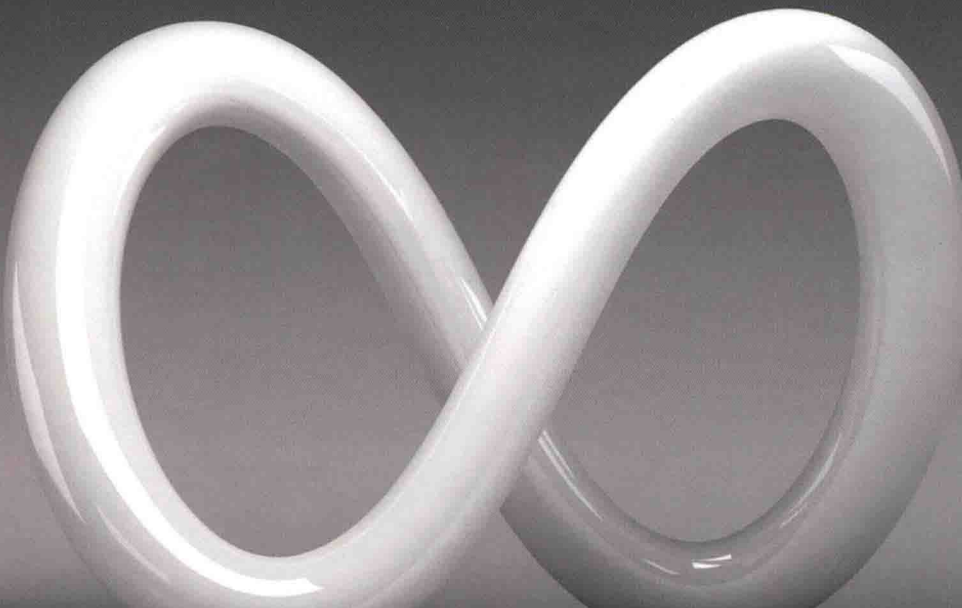
An Invitation to Knot Theory

Virtual and Classical

Heather A. Dye



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To my husband Donovan and my children, Rowan and Gareth,
and my family and friends

Preface

This book is an introduction to virtual and classical knot theory. The book is designed to introduce key ideas and provide the background for undergraduate research on knot theory. Suggested readings from research papers are included in each chapter. The goal is for students to experience mathematical research. The proofs are written as simply as possible using combinatorial approaches, equivalence classes and linear algebra so that they are accessible to junior-senior level students.

I enjoy explaining virtual knot theory to undergraduate students. In the combinatorial proofs, I find examples of many of the concepts that we to try to introduce students to in a course on proof writing. Undergraduates have also experienced a high level of success in projects relating to virtual knot theory. For example, in 2003 Anatoly Pregal won 3rd place in the 2003 Intel Science Talent Search for a project entitled "Computation of quandle cocycle invariants." (Quandles are one of the topics in this book.) Virtual knot theory is a field where undergraduates can experience research in mathematics in a way that results in peer reviewed, publishable research. Another wonderful aspect of studying virtual knots is that it is current research. Students could contact many of the researchers mentioned in this text and participate in the mathematical community.

To write this book, I've drawn on resources that include research articles on virtual knot theory and classic texts on knot theory. Virtual links are the central topic in this book; classical links are a subset of virtual links. The virtual link invariants defined in this text are invariants of both classical and virtual knots. Proofs of invariance are written for virtual links and as such apply to classical knots. For this reason, the proofs are sometimes different from the proofs in classical texts. Well-known applications and results for classical knots and links are also covered in the text. For example, the fundamental group of knot is defined in this text, but our main attention is on the knot quandle (of which the fundamental group of a knot is a homomorphic image).

Classical knot theory originated in the 1880s with Lord Kelvin's proposal that atoms were knotted vortices in the ether. The subject grew from there with P. G. Tait's construction knot tables. The Reidemeister moves were introduced in the 1920s by Kurt Reidemeister's in his article "Knotentheorie." The Alexander polynomial invariant was introduced in the 1928 article, "Topological invariants of knots and links," published in the *Transactions of the American Mathematical Society.* The Alexander polynomial was the main invariant of knots until the introduction of the Jones polynomial in 1985. Much work was done in the area of knot theory involving hyperbolic geometry and topology, but a main focus of this text is the normalized f-polynomial which is related by a change of variables to the Jones polynomial. The Jones polynomial was introduced in a 1985 article, "A Polynomial Invariant for Knots via von Neumann Algebras," published in the *Bulletin of the American Mathematical Society* by V. Jones. This polynomial was quickly followed by the HOMFLY-PT polynomial and a skein relation introduced by Louis Kauffman in 1985. The development of this work encompassed a hundred year span.

In comparison, the development of virtual knot theory spans only twenty years. Virtual knot theory was first defined by Louis Kauffman as an extension of classical knot theory in 1996. Kauffman's paper "Virtual knot theory" was first published in the *European Journal* *of Combinatorics* in 1999. Kauffman's paper immediately introduced extensions of the fundamental group and the f-polynomial to virtual knots and links. Topological interpretations of virtual knots as embeddings of links in thickened surfaces modulo isotopy and stablization have followed. In 2003, Greg Kuperberg proved that each virtual link has a unique minimal genus surface in which the virtual link embeds. Additionally, there are other topological interpretations. Virtual knots have also led to the development of different invariants related to the biquandle and quandle structures, as well as finite type invariants of virtual knots. There are deep connections between virtual knots and algebraic and topological structures. However, the combinatorial structure of virtual knots opens the field to the undergraduates. The fun of virtual knots is working on these multiple levels; invariants and results can be developed independently of these connections (although the knowledge of the connections certainly inspires the results).

This book is written for students who have completed the first two years of a mathematics major. I assume that students have completed linear algebra and a three semester calculus sequence. The book is intended as a gentle introduction to the field of virtual knot theory and mathematical research. For this reason, references are formatted somewhat differently from a research monograph — a list of undergraduate-friendly references is given at the end of each chapter. The complete list of references for each chapter is given as an appendix in the back of the book. I've also explicitly written out references and information about original papers in the body of the text.

The text is divided into four sections: an introduction to virtual knots and counted invariants, an introduction to the normalized f-polynomial (Jones polynomial) and other skein invariants, and an introduction to algebraic invariants such as the quandle and biquandle. The last section is a brief introduction to two applications of virtual knots: textiles and quantum computation.

Readers of this text should begin with the introduction to virtual knot theory in Chapter 1. After Chapter 1, readers can continue with the remainder of Part I, or move on to either Part II or III depending on the focus of the course. Each section has a particular focus and builds toward a major result or question in the field of virtual knot theory.

Part I introduces link invariants that are calculated by counting subsets of crossings weighted by their crossing sign. Chapter 2 introduces linking invariants of ordered, oriented link diagrams that are ultimately used to define an invariant of oriented knot diagrams. In Chapter 3, we introduce additional equivalences classes of link diagrams and their motivations. The next chapter examines unknotting numbers and related invariants of virtual link diagrams. These invariants are computed by taking the minimum number in a set of non-negative integers that also possibly contain the improper element ∞. Next, we examine how to construct families of diagrams through various methods. We explore whether or not the crossing and unknotting invariants defined earlier in the book differentiate between members of this family. Students should naturally conclude that the invariants defined so far do not differentiate between many of the diagrams constructed in the last section.

In Part II, we explore the bracket polynomial and the Kauffman-Murasugi-Thistlethwaite theorem. Our focus on is on extending this theorem to virtual knot diagrams. The normalized f-polynomial is introduced in Chapter 6 and a theorem giving a weak bound on the span of the f-polynomial. To improve this bound, we introduce background information about 2-dimensional surfaces and the Euler characteristic in Chapter 7. The focus is on being able to compute the genus of an abstract link diagram constructed from a virtual link diagram. In Chapter 8, we use the information about the genus of a virtual link diagram to improve the bound on f-polynomials for checkerboard colorable virtual link diagrams. In Chapter 9, we introduce cut points and checkerboard framings of virtual link diagrams. Each topic is introduced as needed; for example, students familiar with 2-dimensional surfaces can skip the majority of the chapter on surfaces. Chapter 9 concludes with an extension of the KMT Theorem. Part II concludes with Chapter 10; we introduce specializations of the bracket polynomial. This chapter can also be omitted for time.

In Part III, the focus is on algebraic invariants. The first chapter introduces the quandle, an algebraic structure from which we can construct many knot invariants. Using quandles, we construct the fundamental group, the Alexander polynomial, and the determinant of a virtual link. In the first chapter of this section, we see that tricoloring (a type of Fox coloring) distinguishes Kishino's knot from its flip. This partially answers a question about distinguishing a link diagram from its flip. However, we are not able to obtain a general result about a knot diagram and its symmetries. The biquandle and the Alexander biquandle remedy this issue, and we see that these are very effective invariants of non-classical virtual links and yet vanish on classical knots. Finally, we conclude with a section on Gauss diagrams which allow us to easily compute the crossing weight invariants introduced in Part I and the important concept of parity. We use parity to augment the bracket polynomial and immediately detect Kishino's knot. The section concludes with two snapshots of applications: textiles and quantum computation. The chapters on Gauss diagrams (Chapter 14) and applications (Chapter 15) can either be omitted or read as stand-alone chapters.

Acknowledgments

I would like to thank all the researchers who responded to my questions about their most undergraduate-friendly papers. An incomplete list of people who corresponded with me includes: Scott Carter, Micah Chrisman, Allison Henrich, Aaron Kaestner, Slavik Jablan, Seiichi Kamada, Naoko Kamada, Louis Kauffman, Adam Lowrance, Sam Nelson, and Radmila Sazdanovic. I would also like to give special thanks to Micah Chrisman and Gail Reed for their comments.

In addition, I would like to thank my parents, Harold and Elaine Dye, and my sister, Kathryn Gresh.

About the author

Heather A. Dye is an Associate Professor of Mathematics at McKendree University in Lebanon, Illinois. Her favorite courses to teach are linear algebra, probability, graph theory, and knot theory. She regularly works with Honors students on their senior thesis.

Her research focuses on virtual knot theory. She has published articles on virtual knot theory in the *Journal of Knot Theory and its Ramifications*, *Algebraic and Geometric Topology*, and *Topology and its Applications*. She enjoys organizing special sessions on knot theory at sectional meetings of the American Mathematical Society. She is a member of both the American Mathematical Society and the Mathematical Association of America.

Her favorite aspect of virtual knot theory is the opportunity to work with undergraduates and meet fellow mathematicians.

In her free time, she enjoys spending time with her family and quilting.

Symbol List

$\mathbb{Z}$ The set of integers

$\mathcal{D}_{i,j}(L)$ The difference linking number of components C_i and C_j in L

$\mathcal{E}_{i,j}(F)$ The flat linking number of F

$\mathcal{FR}$ The set of free virtual links

$\mathcal{F}$ The set of flat virtual links

$\mathcal{L}^i_j(L)$ The linking number of component C_i over component C_j in L

$\mathcal{P}_d(D)$ The minimum number of cut points in any framing of the diagram D

$\mathsf{us}(K)$ The unknotting sequence number of K

$\sim$ The equivalence relation on link diagrams

$[x]$ The equivalence class of x

$\mathcal{F}_d$ The set of flat (or free) virtual link diagrams

$\mathcal{P}(D, B)$ The number of cuts points in checkerboard framing B of the oriented diagram D

$\mathsf{vu}(K)$ The virtual unknotting number of K

$\mathsf{w}(c)$ The weight of the crossing c

$\chi(F)$ The Euler characteristic of the surface F

$\langle K \rangle$ The bracket polynomial of K

$\langle K \rangle_a$ The arrow bracket polynomial of K

$\leftrightarrow$ A single move relation between two link diagrams

$\mathcal{K}$ The set of classical knots

$\mathcal{P}(K)$ The minimal number of cut points in any framing of any diagram equivalent to K

$\mathcal{V}$ The set of virtual knots and links

$\mathcal{V}_d$ The set of virtual knot and link diagrams

$\mathcal{W}_a(K)$ The set of crossings in K with weight a

$g(S)$ The genus of the surface S

$\det(K)$ The determinant of K

$\operatorname{sgn}(c)$ The sign of crossing c

$\mathsf{c}(K)$ The crossing number of K

$\mathsf{u}(K)$ The unknotting number of K

$\mathsf{vus}(K)$ The virtual unknotting sequence number

$\mathsf{v}(K)$ The virtual crossing number of K

$\neg P$ The negation of the statement P

$Ar_K(A, K_i)$ The normalized arrow polynomial of K

$K_1 \sharp K_2$ The connected sum of K_1 and K_2

$P \Rightarrow Q$ P implies Q

$P_K(A)$ The normalized parity bracket polynomial of K

$\langle K \rangle_p$ The parity bracket polynomial of K

$\mathcal{C}_a(K)$ The a crossing weight number of the knot K

$\mathcal{I}$ A link invariant

$G(K)$ The virtual genus of a virtual K

$a(s)$ The arrow number of the state s

$AB_K(s, t)$ The generalized Alexander polynomial of K

$ABM(K)$ The Alexander biquandle matrix of K

$BQ(K)$ The knot biquandle of K

$Col_X(K)$ The number of colorings of K with X

$F(K)$ The flat f-polynomial of K

$f_K(A)$ The f-polynomial of K

K^F The flip of K

K^I The inverse of K

K^M The mirror of K

K^S The switch of K

$Q(K)$ The knot quandle of K

R_n The dihedral quandle of order n

s_β The all B state

s_α The all A state

$w(K)$ The writhe of K