Applied differential geometry

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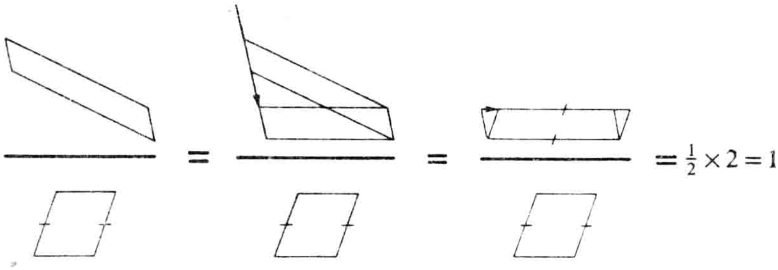
Preface

This is a self-contained introduction to differential geometry and the calculus of differential forms. It is written primarily for physicists. The material complements the usual mathematical methods, which emphasize analysis rather than geometry. The reader is expected to have the standard physics background in mechanics, electrodynamics, and mathematical methods. The mathematically knowledgeable can skip directly to the heart of the book, the calculus of differential forms, in Chapter IV.

This book falls between the usual mathematics and physics texts. On the one hand, proofs are given only when they are especially instructive. On the other hand, definitions, especially of mathematical structures, are given far more carefully than is the usual practice in physics. It is very dangerous to be sloppy in your definitions. I have taken considerable care to give many physical applications and to respect the physical subtleties of these applications. Indeed, my operational rule was to include no mathematics for which I could not produce a useful example. These examples form nearly half the book, and a large part of your learning will take place while reading and thinking about them. I feel that we learn far more from carefully chosen examples than from formal and unnatural deductive reasoning. Most of these examples were originally problems. I wish that I had been left with still more problems for the reader.

I call this a geometric treatment. What do I mean by geometry? One connotation is that of diagrams and pictorial representations. Indeed, I called an early set of notes "The Descriptive Geometry of Tensors." You will find many diagrams here, and I have gone to some effort to make them honest diagrams: Often they can be substituted for the verbal hints that sometimes constitute a proof.



An illustration of the comparison of two areas. No metric is needed, only the construction of parallel lines and the comparison of lengths along a line.

By geometry, however, I mean more than pictures. After all, pictorial methods grow clumsy in spaces of dimension greater than four or five. To be geometric connotes for me an emphasis on the structures themselves, rather than on the formal manipulation of representations, especially algebraic ones.

Examples: To appreciate the distinction between the simplicity of a geometric concept and the complexity of its numerical representation, consider an ellipse in general position. A simple idea, but represented by an appalling mess of numbers.

A determinant can be defined in terms of explicit rules of computation. But geometrically we can better define the determinant as the factor by which a linear transformation changes volumes. Now, to compare the volumes of two parallelopipeds does not require a metric structure. A linear structure is sufficient. For example, a comparison of the areas of two parallelograms is shown in the accompanying diagram. In this geometric view, the determinant of the composition of the two linear transformations (matrix product) is obviously the product of the determinants. To prove this from the algebraic view requires an involved calculation given in Section 27.

The emphasis on the structures themselves rather than on their representations leads us naturally to use the coordinate-free language of modern mathematics.

This modern language makes the foundations of physical models clear and precise. Surprisingly, it also makes the computations clearer. Further, the coordinate-free language turns out to be very easy to illustrate. As this material developed, there was a useful symbiosis between formalism and concrete calculation. The solutions to concrete problems often led to improvements in the formalism. For example, I taught a Jackson-level electrodynamics class using differential forms, and this led to an improved definition of the Hodge star operator. It also forced me to learn and use twisted tensors. Thermodynamics taught me the importance of contact manifolds and affine structures. Books that remain on the formal level treat these important geometric objects briefly if at all.

This emphasis on concrete applications and proper geometric structures helps us avoid the formal symbol manipulations that so often lead to nonsense or fallacious proofs of correct results. [Look at Figure 3.1 in Soper (1976) or the horrible calculus of variations manipulations and mistakes in Goldstein (1959).] Here we will be able to turn most of the infinitesimals commonly seen in physics into the appropriate geometric objects, usually into either rates (tangent vectors) or gradients (differential forms). The distinction between these is lost in the metric-blinded symbol pushing of tensor calculus. Nor will the funny deltas of the calculus of variations with their ad hoc rules of manipulation be found here.

The material of this book grew out of the first quarter of a fairly ordinary general-relativity course. After teaching the course several times, I realized that general relativity as it is usually taught was bad for the student. The introduction of a metric right at the start obscures the geometric structures and would force us into abstract number shuffling. As I put the metric later and later into the course, I discovered examples from an ever-wider range of applications: classical mechanics, dispersive waves, thermodynamics, and so on. After a while I grew embarrassed at pretending that it was still a general-relativity course and, presto, a course on applied differential geometry appeared.

Although most of this material is fairly standard, and by intent the notation is conventional, two departures from common practice should be mentioned. One is the use of twisted tensors. The importance of twisted tensors in physics has been neglected by nearly everyone. There is not even agreement on the name for these objects. Some of these twisted tensors are related to the axial vectors of physics. The second novelty is the use of contact manifolds. These, rather than symplectic manifolds, are the proper setting for most physical theories. Symplectic geometry is important for special situations, primarily time-independent Hamiltonian mechanics.

You might ask, why not learn these fancy methods after the usual tensor calculus? Ordinarily I would agree with a progressive, top–down approach. For the material here it doesn't work. The key idea is that here we are removing structure from our geometric objects, not adding it. To think of a space without a metric or a linear structure is much harder than to think about adding structure to a space, for example, adding a multiplication rule to turn a vector space into a Lie algebra. To work hard to erase what you have just worked hard to learn is frustrating and inefficient. Thus I recommend here the bottom–up approach. You should not view this as the overthrow of all that you have learned, however. Rather, view it as a natural development of vector calculus.

It is impossible for me to recall all the sources for this material. Particularly helpful were Frank Estabrook, Jim Swift, Richard Cushman, Ralph Baierlein, Kim Griest, Hume Feldman, and David Fried, but hold them blameless for my mistakes and idiosyncrasies.

W. L. B.

Glossary of notation

Entries are generally arranged in order of first occurrence in the text.

$\to$	relates source and target sets of a map, as in $f: A \to B$
$\mapsto$	relates a typical source element and its corresponding target element, as in $g: x \mapsto x^3$
$\Rightarrow$	logical implication
$\times$	Cartesian product of two sets; rarely, the 3-space vector cross product
$\circ$	composition of maps
$\{\ \}$	encloses a list of the elements of a set
$,$	partial differentiation, as in $f_{,x}$
$\sim$	equivalence relation
$[b]$	the equivalence class containing the element b
$[a:b:c]$	homogeneous coordinates
$\in$	relates an element to its set, as in $x \in \mathbb{R}$
$\subset$	set inclusion, as in $A \subset B$
V, V^*	a vector space, and its dual
e_i	basis vectors
f^i	dual basis vectors
f'	differentiation of a function with respect to its argument
$Df(u)$	the differential of the function f at the point u
$\cdot$	action of a linear operator (replaces the generally used parentheses); also a missing argument of a function
$\Vert\ \Vert$	norm on a vector space
$\vert\ \vert$	absolute value
A^t	transpose of the matrix A
$D^2f(u)$	second differential of f at .

$\mathbb{R}$	the set of real numbers, also $\mathbb{R}^2$, and so on
$\otimes$	tensor product
$\binom{p}{q}$	the type of tensor, that has p upper indices and q lower indices; a tangent vector is of type $\binom{1}{0}$
$\wedge$	alternating product, sometimes called exterior product
$\bullet$	metric inner product
$\mathcal{G}$	a metric tensor
$\mathcal{E}$	a Euclidean metric tensor
z	the redshift; $(1+z)$ is the ratio of new wavelength to old
ϕ^i	the ith chart map
C^∞	the set of infinitely differentiable functions
S^n	the surface of the sphere in $(n+1)$-dimensional Euclidean space; S^1 is the circle
$T_p(M)$	the set of tangent vectors at the point p of a manifold M
$T_p^*(M)$	the set of 1-forms at the point p of a manifold M, dual to $T_p(M)$
$\dot{\gamma}$	the name for an object that is like the rate of change of γ
$\partial/\partial x$	a basis tangent vector, tangent to the x axis
dx	a basis 1-form, the gradient of the x coordinate function
$X(s)$	when the independent variable x is represented by a function, it is usually written X
$[u, v]$	the Lie bracket of vector fields u and v; rarely, the commutator of two operators
ψ_*	the pushforward map derived from ψ
ψ^*	the pullback map derived from ψ
$\int_\gamma \omega$	the integration of the differential form ω along the curve γ
π	a projection map
$\pi : E \to B$	a fiber bundle with projection map π, total space E, and base space B
$\pi^{-1}(b)$	the set of elements mapped onto b by the map π
T^*M	the cotangent bundle of M
TM	the tangent bundle of M, dual to T^*M
Tf	the tangent map derived from f
CM	the line-element contact bundle of M
C^*M	the hypersurface-element contact bundle of M, not dual to CM
$C(M, n)$	the bundle of n-contact elements of M
F_x	the name for a variable which is like a partial derivative of F
L_b	left translation by the Lie-group element b
$\pounds_w \Omega$	the Lie derivative of Ω by the vector field w

$\boldsymbol{B}$	a 3-vector, rarely used
∇	the 3-vector differential operator del
$\mathcal{I}$	an ideal of differential forms
Λ^r	the set of r-forms
$\lrcorner$	the contraction operator, as in $v \lrcorner \omega$
ϵ^{xyz}	the permutation symbol; can also be written with lower indices
d	exterior derivative
θ	volume element
$\boldsymbol{\theta}$	basis $(n-1)$-forms derived from the volume element θ
$*$	Hodge star operator
$\star$	the special Hodge star in spacetime, used when the 3-space star is also present
#	the sharp operator mapping 1-forms to tangent vectors using a metric
$\vert \; \vert$	used around a set of indices to indicate an ordered summation
$\delta^{\mu\nu}_{\sigma\tau}$	Kronecker delta – may have any number of indices on top, with the same number on the bottom
ω^A	an uppercase index denotes a block index: a string of ordinary indices of unspecified length
(β, Ω)	the representation of a twisted tensor using the ordinary tensor β and an orientation Ω
$\{B\}$	a differential form representing the orientation of the object B
∂M	the boundary of the set M
δ	the differential operator adjoint to d
Δ	the de Rham operator, $d\delta + \delta d$
div	the divergence operator, as in div v
Ω^n	when Ω is a differential form, this indicates the n-fold exterior product
$\Gamma^\mu{}_{\lambda\nu}$	connection components
$\nabla_A B$	the covariant derivative of the vector field B in the A direction
$T(\cdot,\cdot)$	torsion tensor
$R(\cdot,\cdot)\cdot$	curvature tensor
;	covariant derivative, for example, the $\omega_{\mu;\nu}$ are the components of the covariant derivative of the 1-form $\omega_\mu \, dx^\mu$
$\stackrel{*}{=}$	equality that holds only in special coordinates
ω^i	a frame of orthonormal (or pseudoorthonormal) 1-forms
$\omega^i{}_j$	connection 1-forms
Ω_{ij}	curvature 2-forms

Contents

Introduction

No one would try to teach electrodynamics without using vector calculus. It would be crazy to give up so powerful a tool, and it is hard for us to appreciate the resistance to vector calculus shown at the turn of the century. The mathematics of this book can be thought of as the proper generalization of vector calculus, div, grad, curl, and all that, to spaces of higher dimension. The generalization is not obvious. Ordinary vector calculus is misleading because the vector cross product has special properties in three dimensions. This happens because, for $n=3$, n and $\frac{1}{2}n(n-1)$ are equal. It is also important to divorce the formalism from its reliance on a Euclidean metric, or any metric for that matter. Other structures are important, and we must make room for them. Also, a metric allows some accidental identifications that obscure the natural properties of the geometric structures. Similarly, the linear and affine structure of Cartesian spaces should be included only if it is appropriate. The mathematics satisfying these conditions is not classical tensor calculus, but what is called calculus on manifolds and, in particular, the calculus of differential forms.

What physical problems does this calculus address? The basic idea of any calculus is to represent the local behavior of physical objects. Suppose you have a smooth distribution of, say, electric charge. The local behavior is called charge density. A graphical representation of this density is to draw a box, a parallelopiped, that encloses a unit amount of charge (on the average) in the limit where the boxes become smaller than the variations in the density. These boxes have volumes but no particular shapes. What they have is the idea of relative volume: Given two such boxes, we can find the ratio of their volumes. (See the example in the Preface.) Charges come positive and negative, and associated with each

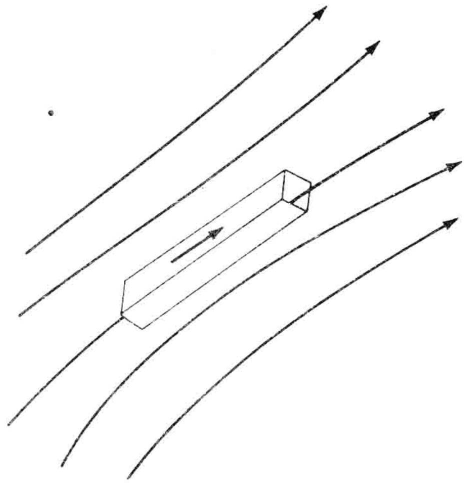


Figure 1. A geometric representation of the current density at a point. The box lies along the current lines and encloses one unit of current in the limit where the box shrinks to nothing. The arrow points in the direction of current flow.

box is a sign. These geometric objects form a one-dimensional vector space at each point.

A more complicated situation would be to have a smooth distribution of electric current. To represent this, draw another box, this one of indefinite length in one dimension. The box should be aligned so that no current flows through the sides, and the cross section is such that it encloses a unit amount of current in the same limit as before. In addition to the shape and alignment, the box now needs an arrow pointing in the direction of current flow (Figure 1). In Chapter IV such geometric objects will be discussed; they are called twisted 2-forms.

Example: A larger current is represented by a box of smaller cross section. The cumulative particle flux in an accelerator is such a current density, and, reasonably enough, it is commonly measured these days in inverse nanobarns.

Such current densities form a three-dimensional vector space. The rules for equivalence, scaling, and addition are easy to discover, and are illustrated in Figures 2, 3, and 4. These geometric objects also describe electric

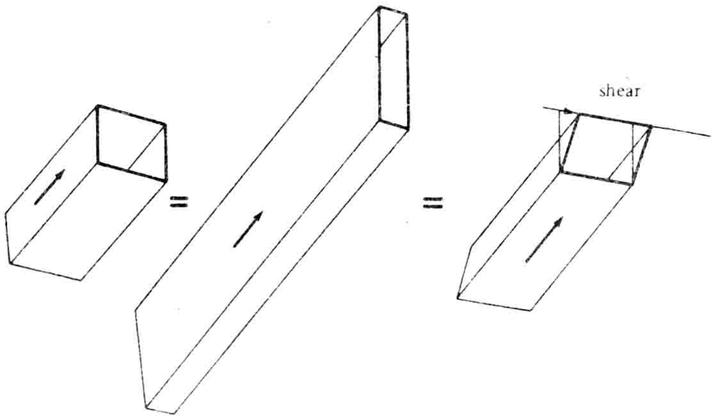


Figure 2. The equivalence of different representations of current density. Each box encloses the same current in the limit.

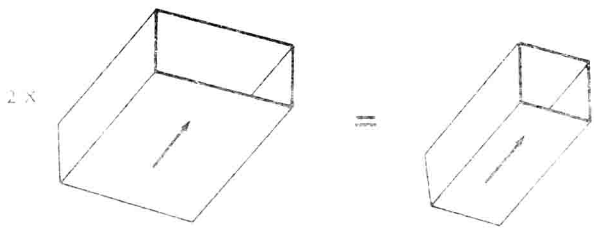


Figure 3. The scaling of current densities. Each encloses the same current.

flux. The field lines appear on positive charges and disappear on negative charges.

Another, related geometric object has a representation given by two parallel planes. Now two of the three dimensions are of indefinite extent. With a convention marking one of these planes as the higher, this geometric object can represent a potential gradient. In these last two geometric objects, we see two different aspects of the electric field: first as the quantity of flux, second as the intensity of the field. This pairing of variables is reminiscent of thermodynamics.

All the geometric objects described so far are types of differential forms: Charge density is of the type called a 3-form, current density is a type of 2-form, and the potential gradient is a 1-form. Each, as you can

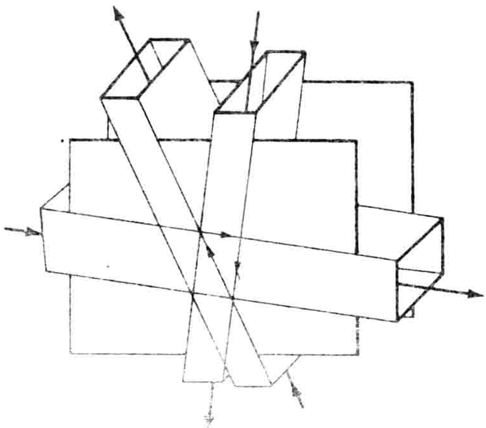


Figure 4. The addition of current densities, shown here in symmetric form. Three current densities that add up to zero are shown. The negative of any one is the sum of the other two.

see, is a natural geometric object in its own right. Although there is some computational convenience to defining, say, 2-forms as objects made up from two 1-forms, this obscures the real meaning of a 2-form. Whenever possible I will try to penetrate the accidental properties inherent in merely convenient representations and constructions, and display the real properties of our geometric objects. For this, pictorial representations and concrete examples will be invaluable.

These geometric objects represent the local behavior of things. Their representations will be in terms of coordinates, but these coordinates will not usually have any intrinsic meaning. Thus their representations must endure under arbitrary smooth changes of the coordinates. Locally these coordinate changes are linear transformations. Note how the constructions given use only objects, such as lines, planes, and volume ratios, that are invariant under linear transformations.

Physical laws relate the behavior of these geometric objects at different points. In electrodynamics, flux lines and current are conserved except where charge density is present or changing. In electrostatics, the electric field can be derived from a potential and has no curl. We will find a natural differential operator, the exterior derivative, with which to express these laws.