Nonlinear Dynamics & Chaos

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Nonlinear Dynamics and Chaos

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Editors

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PREFACE

A summer school on Nonlinear Dynamics and Chaos was held for three weeks from 7th to 25th January 1991 in the Research School of Physical Sciences and Engineering, The Australian National University, Canberra. This school was the fourth in an annual series, organised by the Department of Theoretical Physics. The main aim of the school was to expose participants to the basic concepts and recent developments in nonlinear dynamics in conservative and dissipative systems. Nonlinear dynamics has been developing so rapidly in the past fifteen years or so that much of the material is not available in textbooks. The scope of the topics covered was very broad, including the basic mathematical theory as well as applications in physics, electrical engineering, and even management theory. The physical systems considered were mainly classical but there was some lively discussion of the status of 'quantum chaos'.

The forty five lectures comprising the formal part of the School were given by twelve speakers from within Australia and overseas. Professor Robert Helleman from the University of Houston, Texas, opened with a personal yet very comprehensive introductory overview of the entire field. The final block of lectures, on new developments in the subject of spatio-temporal chaos, was given with great aplomb by Professor Itamar Procaccia from the Weizman Institute in Israel despite his worries about Scud missiles landing on his home town! This book continues the tradition of publishing the lecture notes in a form suitable for long term reference by the participants, and will, we hope, provide a good introduction to the field for other readers. In some cases where lecturers have published reviews elsewhere they have concentrated on newer material. Professor Carl Oberman gave three lectures on nonlinear interactions in plasmas but was prevented by health reasons from completing his manuscript*. We wish him a speedy recovery from his operation.

An innovation in this summer school was the provision of computing facilities. We thank the Computer Services Centre of the Australian National University for their assistance and for allowing us the use of the Computer Services/ANUTECH computer laboratory. There the participants could run several of the excellent computer programs which are now available for interactive visualisation of dynamical systems on DOS PC's, Apple Macintoshes and Unix (Sun) workstations. IBM Australia also kindly lent one of their RISC System/6000 530 advanced Unix workstations, with Silicon Graphics

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* To get an idea of the scope of this big field we refer the reader to many of the articles in *Basic Plasma Physics I and II*, eds. A.A. Geleev and R.N. Sudan (Handbook of Plasma Physics Series, gen. eds. M.N. Rosenbluth and R.Z. Sagdeev, North Holland, Amsterdam, 1983 and 1984), in particular Ch. 5.5, Vol. II, pp. 183–268, by J.A. Krommes.

adaptor board for rapid three dimensional graphics. We thank James Yorke, of the University of Maryland, John Guckenheimer of Cornell University, Bruce Stewart of Brookhaven National Laboratory and Keith Briggs of LaTrobe University for their kind permission to allow us to use their dynamics programs.

The summer school was organised by a committee consisting of R.L. Dewar (Convenor), W.A. Coppel, B.I. Henry and B.A. Robson. Dr G.R.W. Quispel was also on the committee until his move to LaTrobe University. In addition to the support of the Research School of Physical Sciences, the school received sponsorship from the Australian Commonwealth Department of Industry, Technology and Commerce and the Lefebvre Foundation.

We are grateful to Professor John Carver, Director of the Research School of Physical Sciences and Engineering for his interest and support. We are indebted to all the lecturers, the approximately 120 participants, and to the members of our Department for their helpful cooperation during the entire period of the school. We are particularly grateful to Mrs Joan Rowley, Mrs Kay Scott and Ms Martina Landsmann for their considerable assistance in running the school and in preparing the manuscript for this Proceedings. We are grateful to David Sholl for helping prepare the lecture notes of Professor Procaccia, and to Adrienne Fairhall for preparing the first draft of the lecture notes of Dr Quispel and for assistance with the computer laboratory.

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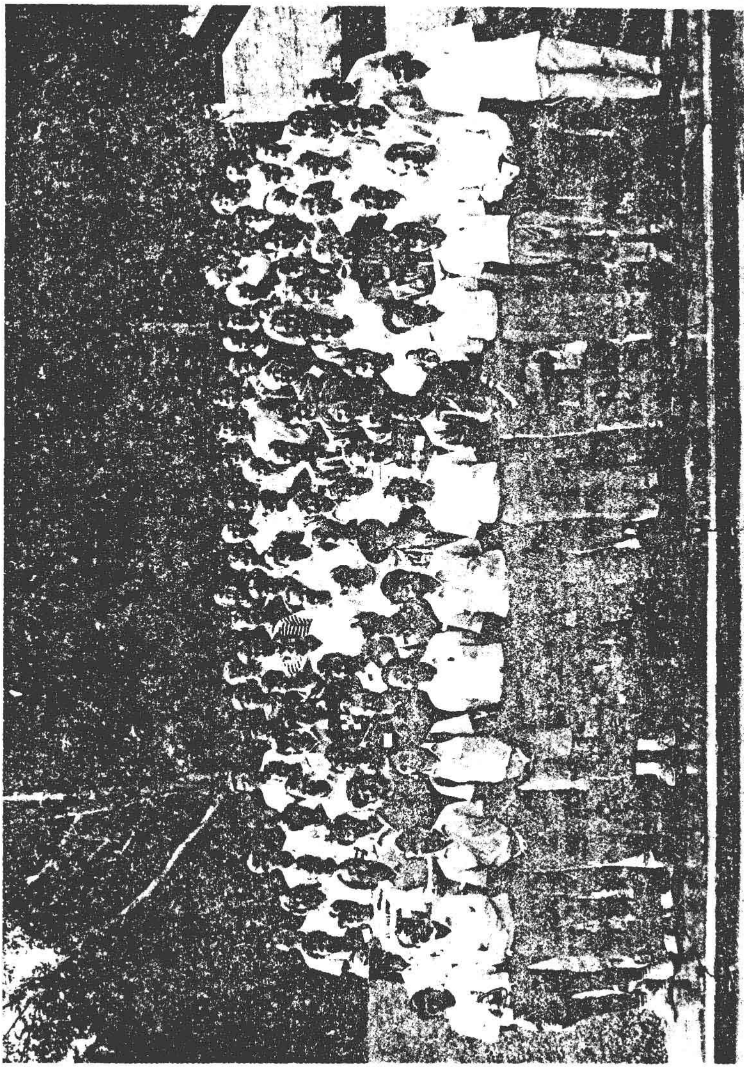

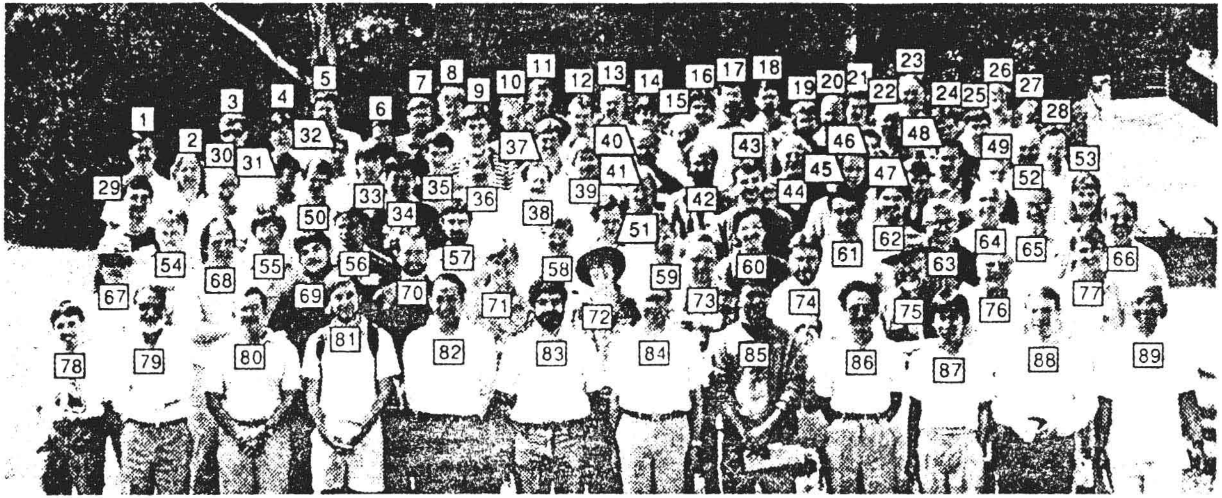

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CONTENTS

CHAOS TOYS

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1. Introduction

In this article I discuss a few *"toy" problems* that are particularly simple and useful for Chaos investigations and lectures. If you desire a more complete introduction into the *"Chaotic Behavior of Deterministic (/Mechanical) Systems"* I refer you to my overview articles, in reference 1 (with 68 pages and 300+ references) and reference 2. Additional reference material and overviews can be found in references 3-9, 17, 19, 20.

The two (related) deterministic toys I play with here, have such (uniformly) *random behavior* that we can trivially investigate the consequences for their Classical Mechanics (conservative and/or dissipative), and their Quantum Mechanics as well as their Statistical Mechanics. For more typical mechanical systems (hence with chaos[1]) this would be very difficult to accomplish due to the intermingling of random and "regular" behavior everywhere in phase space.[1, 3-26]

My toys here consist of one or two differen*ce* equations, i.e. "maps". Such maps are completely *equivalent* to the familiar differen*tial* equations of classical and quantum mechanics, and vice versa.[1,26] Newcomers to Nonlinear Dynamics and Chaos usually tell me that, in their heart, they find this equivalence difficult to accept. For the two maps employed here this equivalence to a mechanics problem has been explicitly established.[2, 3, 4, 6, 9, 20, 26] In general, I remind you, that we think of these maps as "Surface of Section" maps.[1, 3, 4] We plot the consecutive intersection points of the continuous (1-dim.) orbit, of a mechanical system, with a 2-dim. plane in phase space. This device maps one point of that plane to another, etc.[26]

In Section 2 I show how it can be true that a *deterministic* equation has solutions that are *random* in time! I do give *exact* examples of such random solutions for the simplest possible nonlinear problem: a quadratic one-step difference equation in one variable, the famous "Logistic Map".[2, 1, 6, 7, 9] The words *"Chaos"* and/or *"Chaotic behavior"* are defined there. Also it is made plausible that the random behavior of the orbits is due to an (exponentially-) "Sensitive Dependence on the Initial Conditions". As a matter of fact it may be *so* sensitive that we quickly accumulate large differences in orbit behavior when we calculate such orbits on a digital computer (for instance with 16 decimal places) compared with the true orbits. Such (experimental) truncation problems are demonstrated in Section 2.

In Section 3 exact examples of random behavior are given for a simple system that can actually be obtained from *Classical Mechanics*.[20] The system consists of *two* coupled (one-step) difference equations, i.e., a map of the plane to itself. One of the equations is identical to an equation already studied in the previous Section, 2. The system itself is the famous "baker's transformation".[3, 4, 11-14, 20] While the equations in Section 2 provide examples of Chaos in a *'dissipative'* system,[1, 2, 6, 7, 9] the baker's map of Section 3 provides examples of random behavior in an (Energy-) *'conservative'* system.[1-5, 8] Also the equations of Section 3 are reversible in time while the map of Section 2 is *ir*reversible.

In Section 4 I briefly touch on the fashionable problem of *"Quantum Chaos, Is there any?"* [17-21] Basically one quantizes a classical mechanics problem with known (classical-) chaotic behavior and studies the (quantized) time evolution resulting from the Schrödinger Equation, i.e. here for the baker's map of the previous Section 3. The general conclusion is that chaos is as *rare* in quantum mechanics as it is prevalent in classical mechanics. This has stirred up some debate about the classical limit of quantum mechanics.[19, 24-25, 17-21]

In Section 5 the same baker's map, of the previous sections, is used to demonstrate the relation with *Statistical Mechanics*. We address the central problem of *Non-Equilibrium* Statistical Mechanics: How to obtain *irreversible* kinetic equations from the underlying *reversible* mechanical equations of motion. In particular, can we derive the (irreversible) "Approach to Thermodynamic Equilibrium" ("0-th Law of Thermodynamics") for some mechanical systems? The baker's map is so (uniformly) chaotic that this whole program is derived here exactly, and simply.[12-14] Already this half-baked toy problem indicates that, although all this can be derived successfully, there are also some (very) particular initial distributions that will not approach Equilibrium and some (other) kinetic equations that do not approach Equilibrium at all, not from any initial distribution.

I hope that you will find these chaos toys as entertaining and instructive as I do.

2. Exact, Random, Solutions

Consider the simplest nonlinear mapping, i.e. the quadratic difference equation,

$$x_{t+1} = -2x_t + 2x_t^2, \qquad t = \ldots, -1, 0, 1, \ldots \tag{2.1}$$

equivalent to the well known "Logistic Map" (at a = 4).[1,2] It has the exact solution:

$$x_t = \frac{1}{2} + \cos(2\pi\phi_t), \qquad \text{with} \qquad (2.2a)$$

$$\phi_t = \langle[2^t \phi_0]\rangle, \qquad \text{i.e. modulo-1} \qquad (2.2b)$$

for any x_0 ($\in [-\frac{1}{2}, \frac{3}{2}]$), as is easily checked by substitution, where $\langle[\phi]\rangle$ denotes the fractional part of ϕ. This produces the *much simpler equation of motion*,

$$\phi_{t+1} = \langle[2\phi_t]\rangle, \qquad \text{i.e. mod-1} \qquad (2.3)$$

still equivalent to Eq. (2.1) ($x_0 \in [-\frac{1}{2}, \frac{3}{2}]$) . Consider, for a moment, a comparable equation,

$$\phi_{t+1} = \langle[10\,\phi_t]\rangle, \qquad \text{i.e. mod -1} \qquad (2.4)$$

with some arbitrary initial value, e.g. $\phi_0 = 2.360679 \ldots$. The equation of motion, Eq. (2.4), then yields,

$$\begin{aligned} \phi_0 &= 2.360679 \ldots \\ \phi_1 &= 3.60679 \ldots \\ \phi_2 &= 6.0679 \ldots \\ \phi_3 &= 0.679 \ldots, \text{etc.} \end{aligned} \qquad (2.5)$$

Note that the leading digit of ϕ_t simply shifts through all subsequent decimal digits of the initial ϕ_0. Hence, if the digits in ϕ_0 are chosen "randomly" then the ϕ_t moves equally "randomly" in time. It follows from Number Theory that for "almost all" real choices of ϕ_0 these digits are randomly distributed indeed ![10] Of course that is not true for *literally every* choice of ϕ_0 (e.g. not for 0, 0.1, etc.). It also remains true for almost all values of ϕ_0 even if we expand ϕ_0 as a binary number, i.e. in 0's and 1's (rather than as a decimal number as we did in Eq. (2.5)).[10] Returning to our original Eq. (2.3), we now expand ϕ_0, and all ϕ_t's, as binary numbers. Since Eq. (2.3) merely multiplies each number by 2 (mod–1) the (leading binary digit of) ϕ_t simply shifts through all binary digits of ϕ_0, cf. Eqs. (3.2-.4). Hence, for almost all choices of ϕ_0, or equivalently of x_0, the ϕ_t, and x_t move randomly. This is *exact random behavior*.

While the behavior above is very chaotic indeed, the name "chaotic"