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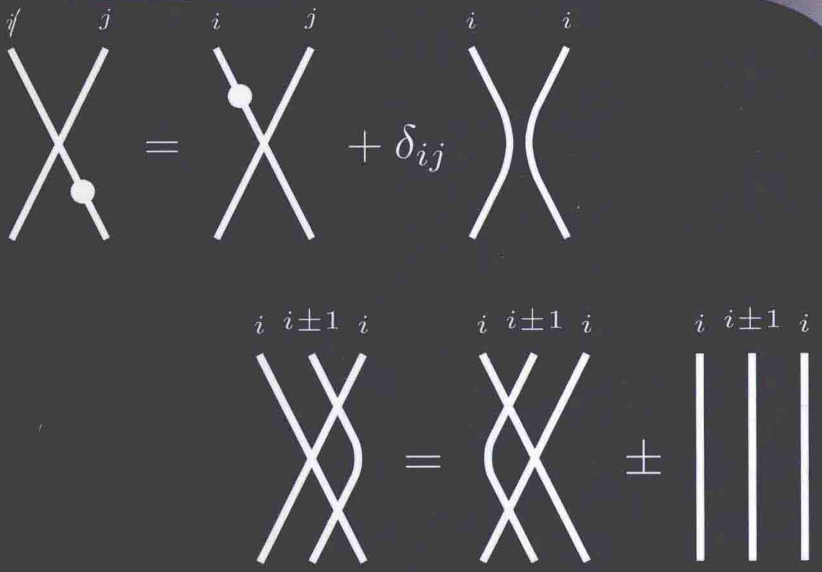

Editors

Wee Teck Gan
Kai Meng Tan

MODULAR REPRESENTATION THEORY OF FINITE AND p-ADIC GROUPS

MODULAR REPRESENTATION THEORY OF FINITE AND *p*-ADIC GROUPS

This volume is an outgrowth of the program *Modular Representation Theory of Finite and p-Adic Groups* held at the Institute for Mathematical Sciences at National University of Singapore during the period of 1–26 April 2013. It contains research works in the areas of modular representation theory of *p*-adic groups and finite groups and their related algebras. The aim of this volume is to provide a bridge — where interactions are rare between researchers from these two areas — by highlighting the latest developments, suggesting potential new research problems, and promoting new collaborations.

It is perhaps one of the few volumes, if not only, which treats such a juxtaposition of diverse topics, emphasizing their common core at the heart of Lie theory.

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FOREWORD

The Institute for Mathematical Sciences (IMS) at the National University of Singapore was established on 1 July 2000. Its mission is to foster mathematical research, both fundamental and multidisciplinary, particularly research that links mathematics to other efforts of human endeavor, and to nurture the growth of mathematical talent and expertise in research scientists, as well as to serve as a platform for research interaction between scientists in Singapore and the international scientific community.

The Institute organizes thematic programs of longer duration and mathematical activities including workshops and public lectures. The program or workshop themes are selected from among areas at the forefront of current research in the mathematical sciences and their applications.

Each volume of the *IMS Lecture Notes Series* is a compendium of papers based on lectures or tutorials delivered at a program/workshop. It brings to the international research community original results or expository articles on a subject of current interest. These volumes also serve as a record of activities that took place at the IMS.

We hope that through the regular publication of these *Lecture Notes* the Institute will achieve, in part, its objective of reaching out to the community of scholars in the promotion of research in the mathematical sciences.

November 2014

Chitat Chong
Wing Keung To
Series Editors

PREFACE

A month-long program on "Modular Representations of Finite and p-Adic Groups" was held during April 2013 at the Institute for Mathematical Sciences (IMS) in Singapore. The program was organised by Joseph Chuang (City University, London, UK), Karin Erdmann (University of Oxford, UK), Wee Teck Gan (NUS), Florian Herzig (Toronto, Canada), Kay Jin Lim (NUS), Alberto Mínguez (Jussieu, France) and Kai Meng Tan (NUS). The goal of the program is to bring together leading researchers in the areas of modular representation theory of finite groups and p-adic groups to discuss the latest developments in the field, chart out new directions for research and explore possible collaboration. It was well attended by about 50 participants worldwide.

By "modular representations", one means the representations of a group G on a vector space over a field of nonzero characteristic ℓ, when ℓ divides the (pro-)order of G. It is with such ℓ-modular representations that the program is concerned. The groups G which feature in the program include the finite groups of Lie type, their Weyl groups and related Lie or Hecke algebras, as well as the reductive p-adic groups which are infinite topological groups that play an important role in the Langlands program and number theory. While the modular representation theory of finite groups have been pursued for about a hundred years, the case of p-adic groups is a relatively young field, where it began its life about 20 years ago. There is so far relatively little interaction between the two areas and it is the hope of the program to stimulate such interactions.

The program began with six short lecture series (aka tutorials) given by leading experts, followed by a two-week conference. The notes of five of these tutorials are collected in this volume. The chapter by M. Cabanes introduces the finite groups of Lie type and discusses recent results and outstanding conjectures about their modular representation theory. The understanding of the representation theory of these finite groups of Lie type provides a crucial starting point for the study of p-adic groups. The chapters by V. Sécherre and F. Herzig (with notes prepared by K. Kozioł) give an exposition of the state-of-the-art in the ℓ-modular representation theory of p-adic groups when $\ell \neq p$ and $\ell = p$, respectively. They convey the exciting developments in the ℓ-modular and p-modular Langlands program. Finally, the chapters by A. Kleshchev and A. Mathas touch upon the representation theory of the increasingly popular and important Khovanov–Lauda–Rouquier (KLR) algebras. Specifically, Kleshchev discusses the basic representation theory of these KLR algebras and their homological properties, whereas the chapter by Mathas is devoted to the emerging graded representation theory in the special case of type A. These KLR algebras arise from the introduction of the idea of "categorification" in representation theory. The last tutorial, given by R. Rouquier, is an introduction to this revolutionary idea of "categorification". Unfortunately, due to circumstances beyond our control, the content of Rouquier's talks is not included in this volume.

We take this opportunity to thank all the participants of the program for contributing to its success. We especially want to thank the six tutorial lecturers for their excellent presentations and for taking the time and effort to write up their notes for publication. We hope that the chapters contained in this volume will serve as a useful reference for researchers and students in these and related areas.

Finally, we would like to convey our deepest appreciation to IMS for generously and graciously providing both financial and administrative support for our program, without which the program would not have been successful.

November 2014

Wee Teck Gan and Kai Meng Tan
National University of Singapore
Singapore
Volume Editors

CONTENTS

MODULAR REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

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We report on the main results about linear representations of finite reductive groups or finite groups of Lie type. Following the historical order, we comment on representations in the defining characteristic, ordinary characters and representations in non-defining characteristic.

Contents

INTRODUCTION

This survey deals with linear representations of a class of finite groups strongly related to linear algebra itself, finite reductive groups. Finite reductive groups also provide almost all finite simple groups and their essential central extensions. They are therefore omnipresent, explicitly or not, in finite group theory.

Finite reductive groups are finite analogues of reductive algebraic groups whose structure is in turn close to that of Lie groups. This also explains the terminology "groups of Lie type" which is often used. Let us mention also the term "Chevalley groups", which pays tribute to Chevalley's construction of those finite groups from integral forms of semi-simple complex Lie algebras [Ch55]. The later approach is also described in [Cart1], though the prevailing approach is now to see them as fixed point subgroups $\mathbf{G}^F$ under a Frobenius endomorphism $F\colon \mathbf{G} \to \mathbf{G}$ of some algebraic group $\mathbf{G}$, an approach due to Steinberg and allowing to take advantage of the geometry of algebraic groups.

The pace of this survey is intended as quite slow, giving details necessary to understand most definitions. This should suit beginners or more experienced readers from other branches of representation or group theory. We tried to comment on some examples, mainly in type A (general linear groups, finite or not), and we strongly recommend Bonnafé's book [Bn] where much of our matter, and more, is thoroughly explored for SL_2.

In Part I, we describe roughly the main features of representation theory, that is mainly the study of module categories for rings that are mostly finite groups algebras or close analogues. This also leads to the study of associated categories, most famously the derived and the homotopic categories.

We did not comment on the already spectacularly successful methods of categorification, referring instead to Rouquier's series of talks.

Part II comments on the constructions and structure of those groups both from the elementary point of view of split BN-pairs and of algebraic groups.

Part III deals with linear representations most naturally associated with finite reductive groups since performed over the field $\overline{\mathbb{F}}_p$ defining the ambient algebraic group. The classical theorems of Chevalley and Steinberg are supplemented by more recent contributions by Lusztig, though it seems that some questions remain as mysterious as they were 25 years ago.

Part IV, by far the longest, comments on both the theory by Deligne-Lusztig leading to a very precise description of ordinary characters of finite groups of Lie type and more recent investigations on representations in characteristics different from p. The first is presented in many references (see [Sri], [DiMi], [Cart2]) and we quickly recall the main results. The second part was largely initiated by the papers of Fong-Srinivasan ([FoSr82], [FoSr89]) relating ordinary characters of finite classical groups $\mathbf{G}^F$ with blocks of $\overline{\mathbb{F}}_\ell$ ($\ell \neq p$). Many contributions followed by Dipper (decomposition numbers [Dip85a], [Dip85b]), and Geck-Hiss-Malle (ℓ-modular Harish Chandra theory [GeHiMa94], [GeHiMa96]). Broué in [Br90] gave several conjectures, the one on Jordan decomposition of characters being later proved by Bonnafé-Rouquier (see [BnRo03]). We report on all those subjects, which leads us to many theorems on ℓ-blocks, decomposition numbers and modular Harish Chandra series.

I. MODULAR REPRESENTATIONS

We refer to the first chapter of [Be] or the short book [Sch] for most of the information we need.

1. Representations

The framework is the one of representations of non commutative rings A that are mainly finite dimensional algebras over a commutative field K. The finite dimensional representations can be seen as objects of the category A-**mod** of finitely generated A-modules. It may also be that A is an $\mathcal{O}$-free algebra of finite rank over a local ring $\mathcal{O}$, in which case we require that the objects of A-**mod** to be $\mathcal{O}$-free of finite rank.

We do not recall here the terminology of *simple, indecomposable,* or *projective* A-modules. The *Jacobson radical* of A is denoted by $J(A)$, the group of invertible elements of A is denoted by $A^\times$. Simple modules are equivalently called *irreducible representations*, and $\mathrm{Irr}(A)$ denotes their set of isomorphism types. When A is over a field K, we denote by $\mathrm{K}_0(A)$ the *Grothendieck ring* of A, which may be seen as the commutative group $\mathbb{Z}\mathrm{Irr}(A)$ endowed with the multiplication induced by tensor product of modules over K. The multiplicity of a simple A-module S as a composition factor in the Jordan-Hölder series of an A-module M is denoted by $[M : S]$.

We also use the notion of *blocks* as minimal indecomposable two-sided ideal direct summands of A. Note that this also makes sense when A is an $\mathcal{O}$-free algebra of finite rank over a local ring $\mathcal{O}$.

Our modules are left modules but we need also the notion of *bimodule*. It is defined as follows. We have two rings A, B, and T is an A-module on the left, and a B-module on the right, with both actions commuting, that is $a(tb) = (at)b$ for any $a \in A$, $t \in T$, $b \in B$. This of course coincides with the notion of an $A \times B^{\mathrm{opp}}$-module where B^{opp} is the opposite ring of B.

An object T of $A \times B^{\mathrm{opp}}$-**mod** defines a functor B-**mod** $\to A$-**mod** by $M \mapsto T \otimes_B M$. The notion of a *Morita equivalence* is a typical example of such a functor (see [Be] Sect. 2.2).

In connection with representations of finite groups, one may be led to consider the category $\mathrm{C}^{\mathrm{b}}(A)$ of bounded complexes $\dots M^i \xrightarrow{\partial^i} M^{i+1} \dots$ of A-modules ($\partial^{i+1}\partial^i = 0$ for all $i \in \mathbb{Z}$, and only finitely many i's are such that $M^i \neq \{0\}$) and the related *homotopic category* $\mathrm{Ho}^{\mathrm{b}}(A)$.

2. Group Algebras

We are mostly interested in the cases where A is a *group algebra* RG of a (multiplicative) finite group G over a commutative ring R. Recall that this is the R-free R-module of all sums $\sum_{g\in G} r_g g$ (for $(r_g)_{g\in G}$ any family of elements of R indexed by G) with R-bilinear multiplication extending the one of G. For us, an R-linear representation of G is an R-free RG-module M of finite rank or equivalently a group morphism $G \to \mathrm{GL}_n(R)$.

When $H \leq G$ is a subgroup, restriction Res^G_H has an adjoint on both sides Ind^G_H which is the functor RH-**mod** $\to RG$-**mod** associated with RG seen as a bimodule with translation actions of G on the left and H on the right, i.e. $\mathrm{Ind}^G_H(N) = RG \otimes_{RH} N$ whenever N is an RH-module.

When $R = K$ is a field where $|G|$ inverts and with primitive $|G|$-th roots of 1, then KG is a split semi-simple algebra $\cong \prod_i \mathrm{Mat}_{d_i}(K)$. If moreover the characteristic of K is 0, then the isomorphism type of a KG-module M is given by its trace character $\chi_M \colon G \to K$ sending g to the trace of its action on M. *Irreducible characters* are the ones of simple KG-modules, and one denotes by $\mathrm{Irr}(G)$ the corresponding set of functions on G. This has values in $\mathbb{Z}[\omega_{|G|}]$ where $\omega_{|G|}$ is a primitive $|G|$-th root of 1, it is therefore independent of K. Note that $\mathrm{Irr}(KG) \xrightarrow{\sim} \mathrm{Irr}(G)$. For most of character theory, see [Isa].

When R is a field k of characteristic a prime divisor of $|G|$ with a primitive $|G|_{p'}$-th root of 1, the algebra $kG/J(kG)$ is split semi-simple. Then $\mathrm{Irr}(kG)$ is often denoted as $\mathrm{IBr}(G)$ and identifies with the central functions on $G_{p'}$ (p-regular elements of G) obtained by Brauer's method of lifting p'-roots of 1 in k into an extension of $\mathbb{Z}_p$ (see [Sch] Sect. 3.1).