Norman Steenrod

Princeton Landmarks in Mathematics

The Topology of Fibre Bundles

纤维丛拓扑学

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By

NORMAN STEENROD

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Preface

The recognition of the domain of mathematics called fibre bundles took place in the period 1935–1940. The first general definitions were given by H. Whitney. His work and that of H. Hopf and E. Stiefel demonstrated the importance of the subject for the applications of topology to differential geometry. Since then, some seventy odd papers dealing with bundles have appeared. The subject has attracted general interest, for it contains some of the finest applications of topology to other fields, and gives promise of many more. It also marks a return of algebraic topology to its origin; and, after many years of introspective development, a revitalization of the subject from its roots in the study of classical manifolds.

No exposition of fibre bundles has appeared. The literature is in a state of partial confusion, due mainly to the experimentation with a variety of definitions of "fibre bundle." It has not been clear that any one definition would suffice for all results. The derivations of analogous conclusions from differing hypotheses have produced much overlapping. Many "known" results have not been published. It has been realized that certain standard theorems of topology are special cases of propositions about bundles, but the generalized forms have not been given.

The present treatment is an initial attempt at an organization. It grew out of lectures which I gave at the University of Michigan in 1947, and at Princeton University in 1948. The informed reader will find little here that is essentially new. Only such improvements and fresh applications are made as must accompany any reasonably successful organization.

The book is divided into three parts according to the demands made on the reader's knowledge of topology. The first part presupposes only a minimum of point set theory and closes with two articles dealing with covering spaces and the fundamental group. Part II makes extensive use of the homotopy groups of Hurewicz. Since no treatment of these has appeared in book form, Part II opens with a survey of the subject. Definitions and results are stated in detail; some proofs are given, and others are indicated. In Part III we make use of cohomology theory. Here, again, a survey is required because the standard treatments do not include the generalized form we must use. A reader who is familiar with the elements of homology theory will have little difficulty.

I must acknowledge my gratitude to Professor Sze-tsen Hu and Dr. R. L. Taylor who read the manuscript and suggested many improvements.

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NORMAN STEENROD

May, 1950
Princeton University

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THE TOPOLOGY OF FIBRE BUNDLES

Part I. The General Theory of Bundles

§1. Introduction

1.1. Provisional definition. A fibre bundle $\mathfrak{B}$ consists, at least, of the following: (i) a topological space B called the *bundle space* (or, simply, bundle), (ii) a topological space X called the *base space*, (iii) a continuous map

$$p\colon \quad B \to X$$

of B onto X called the *projection*, and (iv) a space Y called the *fibre*. The set Y_x, defined by

$$Y_x = p^{-1}(x),$$

is called the *fibre over the point x of X*. It is required that each Y_x be homeomorphic to Y. Finally, for each x of X, there is a neighborhood V of x and a homeomorphism

$$\phi\colon \quad V \times Y \to p^{-1}(V)$$

such that

$$p\phi(x',y) = x' \qquad\qquad x' \in V,\ y \in Y.$$

A *cross-section* of a bundle is a continuous map $f\colon\ X \to B$ such that $pf(x) = x$ for each $x \in X$.

The above definition of bundle is not sufficiently restrictive. A bundle will be required to carry additional structure involving a group G of homeomorphisms of Y called the *group of the bundle*. Before imposing the additional requirements, consideration of a collection of examples will show the need for these. The discussion of these examples will be brief and intuitive; each will be treated later in detail.

1.2. The product bundle. The first example is the *product bundle* or product space $B = X \times Y$. In this case, the projection is given by $p(x,y) = x$. Taking $V = X$ and $\phi =$ the identity, the last condition is fulfilled. The cross-sections of B are just the graphs of maps $X \to Y$. The fibres are, of course, all homeomorphic, however there is a natural unique homeomorphism $Y_x \to Y$ given by $(x,y) \to y$. As will be seen, this is equivalent to the statement that the group G of the bundle consists of the identity alone.

1.3. The Möbius band. The second example is the Möbius band. The base space X is a circle obtained from a line segment L (as indicated in Fig. 1) by identifying its ends. The fibre Y is a line segment. The bundle B is obtained from the product $L \times Y$ by matching the two ends with a twist. The projection $L \times Y \to L$ carries over under this matching into a projection p: $B \to X$. There are numerous cross-sections; any curve as indicated with end points that match provides a cross-section. It is clear that any two cross-sections must agree on at least one point. There is no natural unique homeomorphism of Y_x

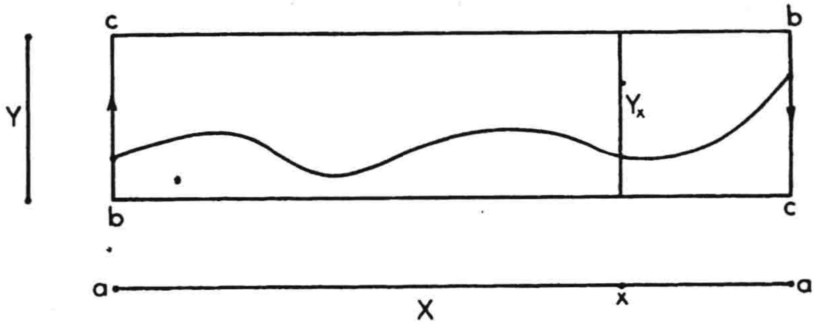

FIG. 1.

with Y. However there are two such which differ by the map g of Y on itself obtained by reflecting in its midpoint. In this case the group G is the cyclic group of order 2 generated by g.

1.4. The Klein bottle. The third example is the Klein bottle. The preceding construction is modified by replacing the fibre by a circle (Fig. 2). The ends of the cylinder $L \times Y$ are identified, as indicated, by reflecting in the diameter de. Again, the group G, is the cyclic group

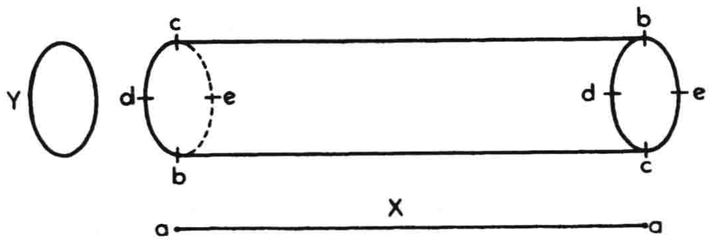

FIG. 2.

of order 2 generated by this reflection. (It is impossible to visualize this example in complete detail since the Klein bottle cannot be imbedded topologically in euclidean 3-space.)

1.5. The twisted torus. The fourth example, we will call the *twisted torus*. The construction is the same as for the Klein bottle except that reflection in the diameter de is replaced by reflection in the center of the circle (or rotation through 180°). As before, the group G is cyclic of order 2. In contrast to the preceding two examples, this bundle is homeomorphic to the product space $X \times Y$ and in such a way as to preserve fibres. However to achieve this one must use homeomorphisms $Y \to Y_x$ other than the two natural ones. But they need not differ from these by more than rotations of Y. This behavior is expressed by saying that the twisted torus is not a product bundle, but it is equivalent to one in the full group of rotations of Y.

1.6. Covering spaces. A covering space B of a space X is another example of a bundle. The projection p: $B \to X$ is the covering map. The usual definition of a covering space is the definition of bundle, in §1.1, modified by requiring that each Y_x is a discrete subspace of B, and that ϕ is a homeomorphism of $V \times Y_x$ with $p^{-1}(V)$ so that $\phi(x,y) = y$. If, in addition, it is supposed that X is arcwise connected, motion of a point x along a curve C in X from x_1 to x_2 can be covered by a continuous motion of Y_x in B from Y_{x_1} to Y_{x_2}. Choosing a base point x_0, each Y_x can be put in 1-1 correspondence with $Y = Y_{x_0}$ using a curve in X. This correspondence depends only on the homotopy class of the curve. Considering the action on Y of closed curves from x_0 to x_0, the fundamental group $\pi_1(X)$ appears as a group of permutations on Y. Any two correspondences of Y_x with Y differ by a permutation corresponding to an element of $\pi_1(X)$. Thus, *for covering spaces, the group of the bundle is a factor group of the fundamental group of the base space.*

1.7. Coset spaces. Another example of a bundle is a Lie group B operating as a transitive group of transformations on a manifold X. The projection is defined by selecting a point $x_0 \in X$ and defining $p(b) = b(x_0)$. If Y is the subgroup of B which leaves x_0 fixed, then the fibres are just the left cosets of Y in B. There are many natural correspondences $Y \to Y_x$, any $b \in Y_x$ defines one by $y \to b \cdot y$. However any two such $y \to b \cdot y$, $y \to b' \cdot y$ differ by the left translation of Y corresponding to $b^{-1}b'$. Thus the group G of the bundle coincides with the fibre Y and acts on Y by left translations. Finding a cross-section for such a bundle is just the problem of constructing in B a simply-transitive continuous family of transformations.

1.8. The tangent bundle of a manifold. As a final example let X be an n-dimensional differentiable manifold, let B be the set of all tangent vectors at all points of X, and let p assign to each vector its initial point. Then Y_x is the tangent plane at x. It is a linear space. Choosing a single representative Y, linear correspondences $Y_x \to Y$ can be constructed (using chains of coordinate neighborhoods in X), but not uniquely. In this case the group G of the bundle is the full linear group operating on Y. A cross-section here is just a vector field over X. The entire bundle is called the tangent bundle of X.