SEMINAR ON MINIMAL SUBMANIFOLDS

EDITED BY

ENRICO BOMBIERI

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INTRODUCTION

The present volume collects the papers which were presented in the academic year 1979-1980 at the Institute for Advanced Study, in the areas of closed geodesics and minimal surfaces, as part of the activities of a special year in differential geometry and differential equations. Starting with a survey lecture, they have been arranged according to dimension and approach, from classical to that of geometric measure theory.

We wish to extend our sincere thanks to all contributors, particularly for their collaboration in sending their texts and their revisions as well as for their patience in waiting for these notes to appear. Our thanks also to the National Science Foundation for supporting this special year at the Institute for Advanced Study.

ENRICO BOMBIERI

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Seminar On
Minimal Submanifolds

SURVEY LECTURES ON MINIMAL SUBMANIFOLDS

Leon Simon*

Our aim here is to give a general (but necessarily brief) introduction to the theory of minimal submanifolds, including as many examples as possible, and including some discussion of the classical problems (Bernstein's, Plateau's) which have provided much of the motivation for the development of the theory.

Our first task is to discuss a notion of minimal "variety" in a sufficiently general sense to include the various classes of objects (e.g. algebraic varieties in $\mathbf{C}^n$, cones over smooth submanifolds of S^n, "soap film like" minimal surfaces in $\mathbf{R}^3$, branched minimal immersions, least area integral current representatives of homology classes) which arise naturally. This will be done in §2, after some classical introductory discussion of first and second variation in §1. In §3 we present some of the principal classes of examples of minimal varieties. §4 includes a discussion of some of the special properties of minimal submanifolds of $\mathbf{R}^n$. In §5 we give a brief survey of the known interior regularity theory, and in §6 we discuss the classical Bernstein and Plateau problems and the present state of knowledge concerning them. Finally, in §7, we discuss some selected applications of second variation of minimal submanifolds in geometry and topology.

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We would here like to recommend the survey articles [L1], [N1], [O1], [B], which cover topics only touched upon (or not mentioned at all) here. For other general reading we strongly recommend the works [FH1], [GE1], [L3], [L4], [N3], [O2].

In all that follows, N will denote an n-dimensional Riemannian manifold without boundary $(n \geq 2)$, and k is an integer with $1 \leq k < n$. U will always denote an open subset of N. We also often have occasion to consider a "smooth deformation" (i.e. a smooth isotopy) of N which holds everything outside a compact subset of U fixed. To be specific, let $(-1,1) \times N$ be equipped with the product metric, let $\phi : (-1,1)\times N \to N$ be a C^2 map, let $\phi_t : N \to N$ be defined by $\phi_t(x) = \phi(t,x)$ for $(t,x) \in (-1,1) \times N$, suppose ϕ_t is a diffeomorphism of each $t \in (-1,1)$, and suppose there is a compact $K \subset U$ such that

$$\phi_0 = 1_N, \quad \phi_t|_{N\sim K} \equiv 1_{N\sim K}, \quad \phi_t(K) \subset K \qquad \forall t \in (-1,1). \tag{0.1}$$

X will denote the associated initial velocity tangent vector field on N, defined by

$$X_x = \frac{\partial}{\partial t} \phi(t,x)\Big|_{t=0}. \tag{0.2}$$

Of course, given any C^1 vector field on N with compact support in U, there always exists ϕ as above such that (0.2) holds.

§1. *First and second variation*

We begin classically, with M denoting an embedded (but not necessarily properly embedded, oriented or complete) k-dimensional submanifold of N such that the k-dimensional volume $\mathcal{H}^k(M \cap K)$* of $M \cap K$ is finite for each compact $K \subset N$.

When U is such that $\mathcal{H}^k(M \cap U) < \infty$ (e.g. if $\overline{U}$ is compact), we say that M is *stationary* (or "minimal") in U if

*Here and subsequently, $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure in N.

(1.1) $$\frac{d}{dt} |\phi_t(M \cap U)|_{t=0} = 0$$

for every ϕ as in (0.1). Here $|\phi_t(M \cap U)|$ denotes the k-dimensional volume of the submanifold $\phi_t(M \cap U)$; that is, $|\phi_t(M \cap U)| = \mathcal{H}^k(\phi_t(M \cap U))$.

M is said to be *stable* in U if it is stationary in U and if

(1.2) $$\frac{d^2}{dt^2} |\phi_t(M \cap U)|_{t=0} \geq 0$$

for each ϕ_t as in (0.1).

We say that M is stationary (respectively stable) in an arbitrary open set $W \subset N$ if it is stationary (respectively stable) in U for each open U with $\mu(M \cap U) < \infty$ and $U \subset W$.

The quantities appearing on the left of (1.1), (1.2) are called the first and second variation of M with respect to the deformation ϕ. We shall see that the first variation depends only on the initial velocity vector X of (0.2), and is otherwise independent of ϕ.

We now want to explicitly compute the first and second variation. To do this, we need the *area formula* ([FH1, §3.2]), which asserts that

(1.3) $$|\phi_t(M \cap U)| = \int_M J(t,x)\, d\mu ,$$

where, for the moment, $\mu = \mathcal{H}^k$ (k-dimensional Hausdorff measure), and where $J(t,x)$ denotes the Jacobian of the restriction of ϕ_t to M. Thus

(1.4) $$J(t,x) = |d_x\phi_t(\tau_1) \wedge d_x\phi_t(\tau_2) \wedge \cdots \wedge d_x\phi_t(\tau_k)| ,$$

where $d_x\phi_t$ denotes the linear map $T_xN \to T_{\phi_t(x)}N$ between tangent spaces induced by ϕ_t, and where $\tau_1, \cdots, \tau_k$ is any orthonormal basis for T_xM. (T_xM is of course equipped with the Riemannian inner product $\langle\ ,\ \rangle$ of T_xN .)

We thus have

$$\frac{d}{dt}\,|\phi_t(M\cap U)|\Big|_{t=0} = \int_M \frac{\partial}{\partial t}\,J(t,x)\Big|_{t=0}\,d\mu$$

and

$$\frac{d^2}{dt^2}\,|\phi_t(M\cap U)|\Big|_{t=0} = \int_M \frac{\partial^2}{\partial t^2}\,J(t,x)\Big|_{t=0}\,d\mu\ ,$$

and direct computation (using (1.4)) shows (see for example the appendix of [SL1] for details) that

$$\text{(1.5)}\qquad \frac{d}{dt}\,|\phi_t(M\cap U)|\Big|_{t=0} = \int_M \mathrm{div}_M X\,d\mu$$

$$\text{(1.6)}\qquad \frac{d^2}{dt^2}\,|\phi_t(M\cap U)|\Big|_{t=0} = \int_M \Big\{\mathrm{div}_M Z + (\mathrm{div}_M X)^2 + \sum_{i=1}^{k} |(\nabla_{\tau_i} X)^{\perp}|^2$$

$$- \sum_{i,j=1}^{k} \langle \tau_i, \nabla_{\tau_j} X\rangle\langle \tau_j, \nabla_{\tau_i} X\rangle - \sum_{i=1}^{k} \langle R(\tau_i, X)X, \tau_i\rangle\Big\}\,d\mu\ .$$

Here X, τ_i are as in (0.2) and (1.4) respectively, $Z_x = \ddot{\phi}_t(x)|_{t=0}$, $x \in M$, $\mathrm{div}_M X = \sum_{i=1}^{k} \langle \tau_i, \nabla_{\tau_i} X\rangle$, ∇_{τ_i} denotes covariant differentiation in N with respect to τ_i, $Y^{\perp}$ (for any $Y \in T_x N$) denotes the orthogonal projection of Y onto $(T_x M)^{\perp}$, and R denotes the Riemannian curvature tensor of N, defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

From (1.5), (1.6) we thus see that M is stationary in U (whether or not $\bar{U}$ is compact) if and only if

$$\text{(1.7)}\qquad \int_M \mathrm{div}_M X\,d\mu = 0$$

for every smooth vector field X on N with compact support in U, and M is stable in U if and only if (1.7) holds and

$$(1.8)\quad \int_M \Big\{(\mathrm{div}_M X)^2 + \sum_{i=1}^{k} |(\nabla_{\tau_i} X)^{\perp}|^2 - \sum_{i,j=1}^{k} <\tau_i, \nabla_{\tau_j} X><\tau_j, \nabla_{\tau_i} X> - \sum_{i=1}^{k} <R(\tau_i, X)X, \tau_i>\Big\} d\mu \ge 0$$

for every such X.

To examine more closely the meaning of these definitions, at each point of M we write $X = X^T + X^{\perp}$, with $X^{\perp}$ denoting the part of X normal to M. Thus, letting $\nu^{k+1}, \cdots, \nu^n$ be a locally defined orthonormal set of vector fields normal to M, we have $X^{\perp} = \sum_{\alpha=k+1}^{n} <\nu^{\alpha}, X>\nu^{\alpha}$, and hence (since $<\tau_i, \nu^{\alpha}> = 0$),

$$(1.9)\qquad \mathrm{div}_M X^{\perp} = \sum_{\alpha=k+1}^{n} <\nu^{\alpha}, X> \mathrm{div}_M\, \nu^{\alpha}.$$

Now

$$\mathrm{div}_M\, \nu^{\alpha} = \sum_{i=1}^{k} <\tau_i, \nabla_{\tau_i} \nu^{\alpha}> = -<\nu^{\alpha}, \sum_{i=1}^{k} B(\tau_i, \tau_i)>,$$

where B denotes the second fundamental form of M, defined on $T_xM \times T_xM$ and taking values in $(T_xM)^{\perp}$ according to

$$(1.10)\qquad B(\xi, \eta) = -\sum_{\alpha=k+1}^{n} <\xi, \nabla_{\eta} \nu^{\alpha}>\nu^{\alpha}, \quad \xi, \eta \in T_xM$$

$$(= B(\eta, \xi)).$$

An alternative definition is

$$(1.10)'\qquad B(\xi, \eta) = (\nabla_{\xi} Y)^{\perp},$$

where Y is any smooth extension of η. The *trace* of B is called the *mean curvature* vector $\mathbf{H}$ of M; thus

$$(1.11) \qquad \mathbf{H}(x) = \sum_{i=1}^{k} B(\tau_i, \tau_i) ,$$

with τ_i as in (1.4).

It goes without saying (and is easily checked) that all these definitions are independent of the particular choice of $\tau_1, \cdots, \tau_k$, $\nu^{k+1}, \cdots, \nu^n$.

We now have from (1.9), (1.10), and (1.11) that

$$(1.12) \qquad \begin{aligned} \operatorname{div}_M X^\perp &= - \sum_{\alpha=k+1}^{n} <\nu^\alpha, X><\nu^\alpha, \mathbf{H}> \\ &= - <X, \mathbf{H}> , \end{aligned}$$

and hence

$$\int_M \operatorname{div}_M X \, d\mu = \int_M \operatorname{div}_M X^T \, d\mu - \int_M <X, \mathbf{H}> d\mu .$$

To go further, we assume that M is actually compact with smooth boundary ∂M (possibly empty); then the classical divergence theorem tells us (since X^T is by definition a tangent vector field on M) that $\int_M \operatorname{div}_M X^T \, d\mu = -\int_{\partial M} <X, \eta> d\mathcal{H}^{k-1}$, where η denotes the co-normal of ∂M. Thus η is tangent to M, normal to ∂M, and points *into* M. Putting these facts together, we thus have

$$(1.13) \qquad \int \operatorname{div}_M X \, d\mu = - \int_{\partial M} <X, \eta> d\mathcal{H}^{k-1} - \int_M <X, \mathbf{H}> d\mu ,$$

and we see that (in case M is smooth, compact) that M is stationary in U if and only if

$$(1.14) \qquad \partial M \cap U = \emptyset \quad \text{and} \quad \mathbf{H} \equiv 0 \quad \text{on} \quad M \cap U .$$

This is of course a well-known classical characterization of stationary submanifolds. It also explains why the word "minimal" is used in the classical setting; one can in fact check that the following lemma holds.

LEMMA 1.1. *Suppose* M *is smooth, compact, and* $\xi \in M \sim \partial M$. *Then* $\mathbf{H} \equiv 0$ *in a neighborhood of* ξ *implies that there is a (small) open* U *containing* ξ *such that*

$$(1.15) \qquad |M \cap U| \leq |\phi_t(M \cap U)|$$

for all sufficiently small t, *whenever* ϕ *is as in* (0.1).

We emphasize that this is in general false if M is allowed to have singularities. (In §2 below we *shall* allow M to have singularities.)

For a brief sketch of the proof of (1.15), we first suppose that N is represented (locally, near ξ, via a coordinate chart) as a submanifold of $\mathbf{R}^n$ in such a way that the point ξ corresponds to $0 \in \mathbf{R}^n$. Then, selecting suitable coordinate axes in $\mathbf{R}^n$ and supposing that U has been chosen sufficiently small, we can write

$$(1.16) \qquad |\phi_t(M \cap U)| = \int_\Omega F(x, u_t, Du_t)dx ,$$

for all sufficiently small t. Here $\Omega \subset \mathbf{R}^k$, $u_t = (u_t^{k+1}, \cdots, u_t^n) : \Omega \to \mathbf{R}^{n-k}$ is such that $Du_0(0) = 0$ and such that graph u_t represents $\phi_t(M \cap U)$, and $F(x,z,p)$ is the "non-parametric k-dimensional area integrand" associated with our local coordinate representation of N. $F(x,z,p)$ is a C^2 function of $x \in \Omega$, $z = (z^{k+1}, \cdots, z^n) \in \mathbf{R}^{n-k}$, and $p = (p_i^\alpha)_{\substack{\alpha=k+1,\cdots,n \\ i=1,\cdots,k}} \in \mathbf{R}^{k(n-k)}$, and it satisfies the uniform convexity condition

$$(1.17)\qquad \sum_{i,j=1}^{k}\ \sum_{\alpha,\beta=k+1}^{n} \frac{\partial^2 F}{\partial p_i^\alpha \partial p_j^\beta}(x,z,p)\xi_i^\alpha \xi_j^\beta \geq c \sum_{i=1}^{k}\ \sum_{\alpha=k+1}^{n} (\xi_i^\alpha)^2$$

for $|p|$, $|z|$ sufficiently small. (In the codimension 1 case, when $n-k=1$, $F(x,z,p)$ is in fact convex in p for all p, but this is not so in case $n-k\geq 2$.)

Now, by virtue of the fact that $\frac{d}{dt}|\phi_t(M\cap U)|_{t=0}=0$, we have the identity

$$|\phi_t(M\cap U)| - |M\cap U| = \frac{1}{2}\frac{d^2}{dt^2}|\phi_t(M\cap U)|_{t=\theta}$$

for some θ between 0 and t. Direct computation now shows that $\frac{d^2}{dt^2}|\phi_t(M\cap U)|\geq 0$ for sufficiently small t (provided U has originally been chosen small enough). (In checking this we use differentiation under the integral in (1.16); one needs to use the convexity (1.17) and also the Poincaré inequality $\int_\Omega \psi^2 \leq c(\text{diam}\,\Omega)^2\int_\Omega |D\psi|^2$, $\psi\in C_0^1(\Omega)$.)

In case M is smooth, compact, stationary in U, and $\partial M\cap U=\emptyset$, we can obtain a somewhat more compact expression for the second variation of M. Indeed if $I(X)$ represents the expression on the right of (1.6) (with first term deleted by virtue of the fact that M is stationary in U), then

$$(1.18)\qquad I(X) = \int_M \Big(\sum_{i=1}^{k} |(\nabla_{\tau_i}X^\perp)^\perp|^2 - \sum_{i,j=1}^{k} \langle X, B(\tau_i,\tau_j)\rangle^2 - \sum_{i=1}^{k}\langle R(\tau_i,X^\perp)X^\perp,\tau_i\rangle\Big)\,d\mu\,.$$

In checking this we first show that $I(X)=I(X^\perp)$ and then use the fact that $\langle\tau_j,\nabla_{\tau_i}X^\perp\rangle = \sum_{\alpha=k+1}^{n}\langle\tau_j,\nabla_{\tau_i}\nu^\alpha\rangle\langle X,\nu^\alpha\rangle = -\langle B(\tau_i,\tau_j),X\rangle$ (ν^α, B as in (1.9), (1.10) above). For further details concerning this computation, see for example [SL1]. In the codimension 1 oriented case, we can write $X = \zeta\nu$, where ν is a smooth unit normal and ζ is a scalar function, and the expression for $I(X)$ becomes

$$(1.19) \qquad I(X) = \int_M \left\{ |\nabla\zeta|^2 - \zeta^2 \left(|B|^2 + \sum_{i=1}^{k} \langle R(\tau_i, \nu)\nu, \tau_i \rangle \right) \right\} d\mu ,$$

where $\nabla\zeta$ denotes the gradient (taken in M) of the scalar function ζ. Notice that $\sum_{i=1}^{k} \langle R(\tau_i, \nu)\nu, \tau_i \rangle$ is just $\mathrm{Ric}(\nu, \nu)$, where Ric is the Ricci curvature of N.

We should finally mention the meaning of the terms "stationary in U " and "stable in U " in case M is *immersed* rather than embedded. To do this we can suppose that M_0 is any compact k-dimensional Riemannian manifold (with or without boundary) and let $\psi : M_0 \to N$ be a smooth map (not necessarily an immersion), and let $J(\psi)$ be the Jacobian of ψ. That is, $J(\psi)(x) = \|\Lambda_k(d_x\psi)\|$, where $d_x\psi$ denotes the linear map $T_xM_0 \to T_{\psi(x)}N$ induced by ψ. The area associated with such a ψ is of course defined by

$$(1.20) \qquad A(\psi) = \int_{M_0} J(\psi)\, d\mathcal{H}^k .$$

By a *variation* of ψ in U_0 (U_0 open in M_0) we mean a 1-parameter family $\{\psi_t\}_{t\in(-1,1)}$ of smooth maps of M_0 into N, smoothly varying in t, such that $\psi_0 = \psi$ and such that, for some fixed compact $K \subset U_0$, $\psi_t(x) \equiv \psi(x)$ whenever $x \in M_0 \sim K$ and $|t| < 1$. Then ψ is said to be *stationary* in U_0 if $\frac{d}{dt} A(\psi_t)|_{t=0} = 0$ for every such variation of ψ, and ψ is said to be *stable* in U_0 if it is stationary in U_0 and if $\frac{d^2}{dt^2} A(\psi_t)|_{t=0} \geq 0$ for every such variation. If we define $X_\xi = \frac{\partial}{\partial t} \psi_t(\xi)|_{t=0}$ (so that X is defined on M_0 but takes values in TN) then one can derive expressions for first and second variations (i.e. for $\frac{d}{dt} A(\psi_t)|_{t=0}$ and $\frac{d^2}{dt^2} A(\psi_t)|_{t=0}$) which are the same as the right sides of (1.5), (1.6), but in which the various quantities must now be appropriately interpreted. (Near points $x_0 \in M_0$, where $J(\psi)(x_0) \neq 0$, ψ *embeds* a small neighborhood W of x_0 into N and one computes the formulae for first and second variation by computing the t-derivatives of the relevant Jacobian as before. On the other hand, the points where $J(\psi)(x_0) = 0$ of course contribute nothing to the formulae.) Of course from the previous discussion, it follows that if $U_0 \cap \partial M_0 = \emptyset$, then ψ is stationary in U_0 if and only if ψ locally embeds M_0 as a *zero mean curvature submanifold* near points where $J(\psi) \neq 0$.

§2. k-*varifolds*, k-*currents*

An examination of the discussion in §1 will show that the formulae (1.5), (1.6) (and their derivations) remain valid even in the presence of serious singularities in M. To make this statement precise, we first need to introduce some terminology.

We henceforth let M be a *countably* k-*rectifiable Borel set* in N. That is, M is Borel, $M \subset \bigcup_{j=1}^{\infty} M_j$, where M_j are (open) k-dimensional C^1 submanifolds of N (M_j not necessarily complete nor pairwise disjoint).(*)

We also allow the introduction of a *multiplicity function* of M; specifically, we let μ be the measure on N defined by $d\mu = \theta d\mathcal{H}^k$, where θ is a non-negative locally $\mathcal{H}^k$-summable function (called the *multiplicity function*) on N. (We take θ to be defined on all on N, rather than just on M, for reasons of purely technical convenience; we of course often have $\theta \equiv 0$ on $N \sim M$.)

(*)Modulo sets of $\mathcal{H}^k$-measure zero, this is equivalent to the requirement that M is contained in a countable union of images, under Lipschitz maps, of compact subsets of $\mathbb{R}^k$. (By [FH1, 3.1.16].)