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HARMONIC ANALYSIS ON HILBERT SPACE

Leonard Gross[1)]

Cornell University

§1. INTRODUCTION. A real valued stochastic process $\xi(t,\omega)$ almost all of whose sample functions lie in some linear space $\mathcal{L}$ of functions on an interval I induces on $\mathcal{L}$ in a natural and well known manner a probability measure μ. Specifically μ is induced by the mapping $\omega \to \xi(\cdot,\omega)$ from the probability space into $\mathcal{L}$. Much of the study of a stochastic process is equivalent to a study of the measure μ on a suitably chosen space $\mathcal{L}$. Most interesting stochastic processes for example have regular enough sample functions to be in $L^2(I)$ when I is a finite interval or in $L^2(I,w(t)dt)$ when I is an infinite interval and $w(t)$ is some weight function. In any case $\mathcal{L}$ is usually a locally convex topological linear space and the Fourier transform ϕ of μ defined by $\phi(y) = \int_{\mathcal{L}} \exp(i\langle x,y\rangle)\, d\mu(x)$, which is a function on the dual space $\mathcal{L}^*$, uniquely determines μ when μ is defined on the σ-ring generated by the weakly open sets. The relationship between properties of μ and properties of its Fourier transform have been studied by a number of authors. See for example Prohorov [11], LeCam [9], Getoor [5], Cameron and Donsker [4] and their bibliographies. In this paper we shall prove three theorems which are Hilbert space generalizations of three classical theorems in the harmonic analysis of probability measures. Theorem 2 is a Hilbert space analogue of the Levy continuity theorem. Theorem 3 is a generalization of the Bochner representation theorem for positive definite functions on a Hilbert space H. It asserts

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that a positive definite function ϕ on H is the Fourier transform of a positive finite measure on H if and only if ϕ is continuous at the origin in a topology which will be described later. Mourier [10], Getoor [5], and LeCam [9] also have representation theorems but their conditions are of a very different kind from ours. In addition to positive definiteness Mourier and Getoor place conditions not directly on the function ϕ itself but on the finite dimensional measures obtained by partially inverting ϕ while LeCam's conditions involve linear combinations of ϕ at different points. R. A. Minlos [15] proves a representation theorem for a special kind of nuclear space in which continuity of ϕ at the origin is the only requirement beyond positive definiteness. His result is the closest to ours in this sense. Theorem 4 describes a class of inversion formulae for the Fourier transform of a probability measure on H. Necessary and sufficient conditions are given in this theorem for a particular inversion formula to hold. We thereby extend the domain of validity of some of the inversion formulae developed by Cameron and Donsker in [4], which paper formed the starting point of our investigation. We are indebted to Professor Mark Kac for pointing out to us the relevance of the last named paper to our own work.

A fundamental role will be played by the notion of a weak distribution on a linear space. This somewhat more general notion than that of a measure was first defined and exploited by I. E. Segal. In particular the normal distribution which seems to be a good infinite dimensional substitute for Lebesgue measure will be used extensively.

§2. PRELIMINARIES. We shall summarize first some of the basic results of Prohorov [11] which we shall have occasion to use. Let X be a separable complete metric space and let $\mathfrak{M}$ be the space of positive finite measures defined on the σ-ring generated by the open sets of X. A sequence μ_n of measures in $\mathfrak{M}$ is said to converge weakly to a measure μ in $\mathfrak{M}$ if $\int_X f\, d\mu_n \rightarrow \int_X f\, d\mu$ for every bounded continuous function f on X. It is shown in [11] that the topology thus induced on $\mathfrak{M}$ is equivalent to the topology induced by the following metric on $\mathfrak{M}$: if μ and ν are in $\mathfrak{M}$ then $\rho(\mu,\nu) = \inf\{\epsilon : \mu(F) \leq \nu(F^\epsilon) + \epsilon$ and $\nu(F) \leq \mu(F^\epsilon) + \epsilon$ for all closed $F \subset X\}$ where F^ϵ denotes the open ϵ neighborhood of F. It is also proved in [11] that $\mathfrak{M}$ is complete in this metric and most important is the following theorem.

THEOREM (PROHOROV). A set of measures $\mathfrak{N} \subset \mathfrak{M}$ is precompact if and only if there is a constant M such that $\mu(X) \leq M$ for all μ in $\mathfrak{N}$ and for every $\epsilon > 0$ there exists a compact set K_ϵ such that $\mu(X - K_\epsilon) < \epsilon$ for all μ in $\mathfrak{N}$. Alternatively, $\mathfrak{N}$ is precompact if and only if $\mu(X) \leq M$ for all μ in $\mathfrak{N}$ and for every $\epsilon > 0$ and $\delta > 0$ there exists a finite set F of points in X such that $\mu(X - F^\delta) < \epsilon$ for all μ in $\mathfrak{N}$.

Some of the above results have been extended to non-separable metric spaces in [14]. However in a non-separable Hilbert space it is not clear that the Fourier transform of a measure defined on the σ-ring generated by norm-open sets determines the measure uniquely and it may very well not. There are various assumptions that one might make about the class of measures under consideration to alleviate this and other difficulties. However the resulting theorems seem to be little more than theorems about measures in separable spaces anyway so we shall consider for the most part only separable Hilbert spaces.

By a random variable on a probability space Ω we shall understand an element of the linear space of measurable functions modulo null functions on Ω.

DEFINITION 1. A weak distribution on a topological linear space L is an equivalence class of linear mappings F from the topological dual space L^* to random variables on a probability space (depending on F) where two such mappings F_1 and F_2 are equivalent if for every finite set of vectors $y_1,\ldots,y_k$ in L^* the sets $\{F_i(y_1),\ldots,F_i(y_k)\}$ have the same joint distribution in k space for $i = 1$ or 2.

Here L^* denotes the space of continuous linear functionals on L.

Weak distributions have been investigated in numerous papers by I. E. Segal. Their connections with stochastic processes have been discussed extensively by S. Bochner [2] and [3]. Weak distributions have also been studied as such by R. K. Getoor [5], J. Feldman and the author.

In a finite dimensional space a weak distribution coincides with the notion of a measure i.e. if L is finite dimensional then for any given weak distribution there exists a unique Borel probability measure on L such that the identity map on L^* is a representative of the given weak distribution. We refer the reader to [12] or [7] for further discussion of weak distributions.

The (canonical) normal distribution (with variance parameter one in this paper) on a real Hilbert space H is that unique weak distribution which assigns to each vector y in H^* a normally distributed random variable with mean zero and variance $\|y\|^2$. It follows from the preceding property that the normal distribution carries orthogonal vectors into independent random variables. Some of the theory of integration with respect to a measure can also be carried out with respect to a weak distribution. See [7] and its bibliography in this connection.

We shall assume that the following notions and facts are known to the reader. A <u>tame function</u> on a real Hilbert space H is a function of the form $f(x) = \phi(Px)$ where P is a finite dimensional projection on H and ϕ is a Baire function on the finite dimensional space PH. For such a function we have $f(x) = \psi((x,x_1),\ldots,(x,x_k))$ where $x_1,\ldots,x_k$ is a basis of PH and ψ is a Baire function of k real variables. If F is a representative of a weak distribution then the random variable $f^{\sim} = \psi(F(x_1), \ldots, F(x_k))$ depends only on the function f and the mapping F while integration properties of $f^{\sim}$ such as the integral of $f^{\sim}$, the distribution of $f^{\sim}$, convergence in probability of sequences $f_n^{\sim}$ etc. depend only on f and the f_n and on the weak distribution of which F is a representative. Let us denote by $\mathcal{F}$ the directed set of finite dimensional projections on H directed under inclusion of the ranges. For a given continuous function f on H and a given weak distribution one may consider whether the net $(f \circ P)^{\sim}$ of "tame random variables," where P ranges over the directed set $\mathcal{F}$, converges in probability as P approaches the identity through $\mathcal{F}$. If so then the limit which we shall denote by $f^{\sim}$ is a random variable whose integration properties are determined by the function f and the weak distribution. In [6] and [7] classes of continuous functions are described for which the limit defining the random variable $f^{\sim}$ exists when the weak distribution in question is the normal distribution. We shall need for the most part only a special case of Theorem 1 of [6] and Corollary 5.3 of [7] as follows.

DEFINITION OF THE TOPOLOGY $\mathcal{T}$. $\mathcal{T}$ is defined as the weakest topology on H for which all Hilbert-Schmidt operators are continuous from $\mathcal{T}$ to the strong topology of H. Thus a basic open neighborhood of x_0 is $\{x : \| A(x - x_0) \| < \epsilon\}$ where A is a Hilbert-Schmidt operator.

THEOREM. If a complex valued function f on H is uniformly continuous in the topology $\mathcal{T}$ then $f^{\sim} = \lim \text{ in prob.}_{P \to I} (f \circ P)^{\sim}$ exists with respect to the normal distribution. Furthermore if H is separable and $\{P_j\}$ is any sequence of finite dimensional projections converging strongly to the identity operator then $\lim \text{ in prob.}_{j \to \infty} (f \circ P_j)^{\sim}$ exists and equals $f^{\sim}$.

DEFINITION 2. The Fourier transform (or characteristic functional) of a weak distribution $y \to m(y)$ on a topological linear space L is the function on L^* defined by $\phi(y) = E^{(m)}[\exp(i\, m(y))]$ where $E^{(m)}$ denotes expectation with respect to the weak distribution m.

A rough classification of weak distributions is as follows.

DEFINITION 3. A weak distribution m on a Banach space B is continuous if $m(y_k)$ converges to zero in probability whenever y_k is a sequence converging to zero in norm in B^*. A weak distribution m on a topological linear space L is a measure if there exists a probability measure μ defined on the σ-ring $\mathcal{A}$ generated by the weakly open sets of L such that the identity map on L^* is a representative of m.

A weak distribution which is a measure may and will be identified with the measure μ to which it corresponds in the manner of Definition 3 when the uniqueness of μ is certain. Such uniqueness is easily seen to hold within the class of measures defined on the above mentioned σ-ring $\mathcal{A}$. It is clear that a probability measure on any σ-ring in a Banach space B containing the weakly open sets defines a continuous weak distribution on B.

Concerning the domain of definition of measures on a Banach space B we mention that if B is separable the σ-rings generated by the weakly open sets and the strongly open sets are the same.

The notion of a closed weak distribution may also be defined and has been considered in [13] but will not be studied in this paper. Each of the above smoothness conditions on a weak distribution reflects itself in the smoothness of its characteristic functional. We have for example the following theorem by Bochner [2, Theorem 6] and Getoor [5, Corollary 1 of Theorem 1].

THEOREM. If m is a weak distribution on a Banach space B with characteristic functional ϕ then the following are equivalent

1) ϕ is continuous at the origin of B^*

2) m is continuous

3) ϕ is uniformly continuous on B^*.

In this connection see also Corollary 3.1 of the next section.

DEFINITION. A complex valued function ϕ on a linear space L is positive definite if for every finite set $x_1,\ldots,x_k$ in L and complex numbers $c_1,\ldots,c_k$ there holds $\Sigma_{i,j}\, c_i\, \overline{c}_j\, \phi(x_i - x_j) \geq 0$.

We note here for later use that if B is a Banach space and ϕ is a normalized (i.e. $\phi(0) = 1$) continuous positive definite function on B^* then there exists a weak distribution on B whose characteristic functional is ϕ. This was proved by Getoor [5].

§3. THE CONTINUITY THEOREM AND REPRESENTATION THEOREM. All measures will be defined on the σ-ring generated by the norm-open sets. The following theorem does not require separability.

THEOREM 1. The Fourier transform of a complex valued measure of bounded variation on a real Hilbert space H is uniformly $\mathcal{T}$ continuous on H.

PROOF. If $\phi(y) = \int_H \exp(i(s,y))\, d\mu(s)$ where μ is of bounded variation then $|\phi(y) - \phi(x)| \leq \int_H |\exp(i(s,y)) - \exp(i(s,x))|\, d|\mu|(s) = \int_H |\exp(i(s,y-x)) - 1|\, d|\mu|(s)$. We shall show that the last expression is small when y-x lies in a small $\mathcal{T}$ neighborhood of the origin and for this purpose we may assume that μ is a positive finite measure. Denote by S_r the open sphere of radius r centered at the origin. Then $\mu(S_r) \to \mu(H)$ as $r \to \infty$. Given $\epsilon > 0$ choose r so large that $\mu(S_r) \geq \mu(H) - \epsilon/4$. The bilinear form $[x,y] = \int_{S_r} (s,x)\,(s,y)\,d\mu(s)$ is symmetric, positive and bounded since $|[x,y]| \leq \|x\|\|y\|\; r^2\,\mu(S_r)$. Hence for some positive operator B we have $(Bx,y) = \int_{S_r} (s,x)\,(s,r)\,d\mu(s)$. Furthermore B is a trace class operator since for any finite subset of an orthonormal basis $\{x_\alpha\}$ we have $\Sigma_{j=1}^n (Bx_{\alpha_j}, x_{\alpha_j}) = \int_{S_r} \Sigma_{j=1}^n (s,x_{\alpha_j})^2\,d\mu(s) \leq \int_{S_r} \|s\|^2\,d\mu(s) \leq r^2\,\mu(S_r)$ and hence $\Sigma_\alpha (Bx_\alpha, x_\alpha) \leq r^2\,\mu(S_r)$. Let A be the positive square root of B. Then A is a Hilbert-Schmidt operator. Now choose a positive number δ such that $|\exp(it) - 1| < \epsilon/(4\,\mu(H))$ whenever $|t| < \delta$ and put $\alpha = \delta^2\,\epsilon/8$. We shall show that if $\|A(y-x)\|^2 < \alpha$ then $|\phi(y) - \phi(x)| < \epsilon$. Let $E_\delta = \{s \in S_r : |(s,y-x)| < \delta\}$. Now $\int_{S_r} (s,y-x)^2\,d\mu(s) = \|A(y-x)\|^2 < \alpha$. Hence $\delta^2\,\mu(S_r - E_\delta) < \alpha = \delta^2\,\epsilon/8$ so that $\mu(S_r - E_\delta) < \epsilon/8$. Thus $\int_H |\exp(i(s,y-x)) - 1|\,d\mu(s) \leq 2\int_{H-S_r} d\mu(s) + 2\int_{S_r-E_\delta} d\mu(s) + (\epsilon/(4\,\mu(H)))\int_{E_\delta} d\mu(s) < \epsilon/2 + \epsilon/4 + \epsilon/4 = \epsilon$.

REMARK. In view of the preceding theorem the Fourier transform ϕ of a measure μ of bounded variation defines a random variable with respect to the normal distribution. This random variable will be denoted by $\phi^\sim$. We note also that if ν is a positive real number then $\phi(\nu x)$ is also uniformly $\mathcal{T}$ continuous. The corresponding random variable will be denoted by $\phi(\nu x)^\sim$.

THEOREM 2. Let μ_n be a sequence of probability measures on a real separable Hilbert space H with respective characteristic functionals ϕ_n. Let ϕ be a uniformly $\mathcal{T}$ continuous functional on H such that $\phi(0) = 1$. If μ_n converges weakly to a measure μ whose characteristic functional is ϕ

then ϕ_n converges to ϕ on H and $\phi_n^\sim$ converges to $\phi^\sim$ in probability. Conversely if $\phi_n^\sim$ converges to $\phi^\sim$ in probability then μ_n converges weakly to a probability measure μ with characteristic functional ϕ.

LEMMA 2.1. Let μ be a probability measure on E_k with characteristic function ϕ. Denote by n Gauss measure on E_k with variance one i.e. $dn(x) = (2\pi)^{-k/2} \exp(-\| x \|^2/2)\, dx$. Let A be a linear transformation on E_k and let τ, r and ν be positive numbers. Put $\alpha = n(\{t: \nu \| At \| \leq \tau\})$ and $\beta = 2/((2\pi)^{1/2} \nu r\alpha)$. If $\beta < 1$ then

$$1) \qquad \mu(S_r) \geq \frac{|\alpha^{-1} \int_{\nu\| At \|<\tau} \phi(\nu t)\, dn(t)| - \beta}{1 - \beta}$$

where $S_r = \{x : \| x \| \leq r\}$.

PROOF.

$$\int_{\nu\| At \|\leq\tau} \phi(\nu t)\, dn(t) = (2\pi)^{-k/2} \int_{\nu\| At \|\leq\tau} \phi(\nu t) \exp(-\| t \|^2/2) dt$$

$$= (2\pi)^{-k/2} \int_{\nu\| At \|\leq\tau} \int_{E_n.} \exp[\nu i(t,x) - \| t \|^2/2]\, d\mu(x)\, dt.$$

Interchanging the order of integration and denoting by S_r' the complement of S_r we obtain

$$\left| \int_{\nu\| At \|\leq\tau} \phi(\nu t)\, dn(t) \right|$$

$$\leq (2\pi)^{-k/2} \int_{S_r} \int_{\nu\| At \|\leq\tau} \exp(-\| t \|^2/2)\, dt\, d\mu(x)$$

$$+ \left| \int_{S_r'} \int_{\nu\| At \|\leq\tau} \exp[\nu i(t,x) - \| t \|^2/2]\, dt\, d\mu(x) \right| .$$

The first term on the right of the last inequality is exactly $\alpha\mu(S_r)$. In order to estimate the second term we shall carry out a partial integration. Consider a fixed $x \neq 0$. Let F be the k-1 dimensional subspace of E_k orthogonal to x and denote by R the projection onto F of the (possibly degenerate) ellipsoid $S = \{t : \nu\| At \| \leq \tau\}$. For each t' in R the line $\{sx_1 + t' : -\infty < s < \infty\}$ where $x_1 = x/\| x \|$ intersects the boundary of S in at most two points, say at $a(t')x_1 + t'$ and $b(t')\, x_1 + t'$ where $a(t') \leq b(t')$. If the line does not intersect the boundary of S then it is contained in S and for such a t' put $a(t') = -\infty$ and $b(t') = +\infty$. Then

$$\left| \int_{\nu \| At \| \leq \tau} \exp(\nu i(t,x)) \exp(-\| t \|^2/2) dt \right|$$

$$= \left| \int_R \left(\int_{a(t')}^{b(t')} \exp(i\nu s \| x \|) \exp(-s^2/2) ds \right) \exp(-\| t' \|^2/2)\, dt' \right|$$

$$= \left| \int_R \left(\frac{1}{\nu i \| x \|} \left\{ \left[e^{i\nu s \| x \|} e^{-s^2/2} \right]_a^b + \int_a^b s\, e^{\nu i s \| x \|} e^{-s^2/2} ds \right\} \right) e^{-\| t' \|^2/2} dt' \right|$$

$$\leq \frac{1}{\nu \| x \|} \int_R \left\{ e^{-b^2/2} + e^{-a^2/2} + \int_a^b |s| e^{-s^2/2} ds \right\} e^{-\| t' \|^2/2} dt'$$

$$\leq (1/\nu \| x \|) \int_R \left\{ e^{-b^2/2} + e^{-a^2/2} + \int_0^{|b|} s\, e^{-s^2/2} ds + \int_0^{|a|} s\, e^{-s^2/2} ds \right\} e^{-\| t' \|^2/2} dt'$$

$$\leq (1/\nu \| x \|) \int_R \left\{ e^{-b^2/2} + e^{-a^2/2} + (1 - e^{-b^2/2}) + (1 - e^{-a^2/2}) \right\} e^{-\| t' \|^2/2} dt'$$

$$\leq (2/\nu \| x \|) \int_R e^{-\| t' \|^2/2} dt'$$

$$\leq (2/\nu \| x \|) \int_F e^{-\| t' \|^2/2} dt'$$

$$\leq (2/\nu \| x \|)\, (2\pi)^{(k-1)/2} .$$

Hence

$$\left|\frac{1}{\alpha}\int_{\nu\|At\|\leq\tau}\phi(\nu t)dn(t)\right| \leq \frac{1}{\alpha}(\alpha\mu(S_r) + (2/(2\pi)^{\frac{1}{2}}\nu)\int_{S_r'}\|x\|^{-1}d\mu(x))$$

$$\leq \mu(S_r) + 2/((2\pi)^{1/2}\alpha\nu r)(1-\mu(S_r))$$

$$\leq \mu(S_r)(1-\beta) + \beta$$

from which the assertion of the lemma follows.

REMARK. In case the dimension of E_k is one and A is the identity operator the lemma implies a well known inequality. Thus it is easy to see that $(2\pi)^{1/2}\,\nu\alpha = \int_{|t|\leq\tau} \exp(-t^2/2\nu^2)\,dt \to 2\tau$

as $\nu \to \infty$. Hence if $1/(r\tau) < 1$ then for sufficiently large ν the β of the lemma is less than one and we have $\mu([-r,r]) \geq \lim_{\nu\to\infty}(1-\beta)^{-1}\cdot$

$$\left(\left|(2\pi^{1/2}\nu\alpha)^{-1}\int_{-\tau}^{\tau}\phi(t)\exp(-t^2/2\nu^2)\,dt\right| - \beta\right)$$

$$= (1-(r\tau)^{-1})^{-1}\left(\left|(2\tau)^{-1}\int_{-\tau}^{\tau}\phi(t)\,dt\right| - (r\tau)^{-1}\right).$$

LEMMA 2.2. Let μ be a probability measure on a real separable Hilbert space H. Denote by ϕ its characteristic functional and let A be a Hilbert-Schmidt operator on H. Let τ, ν and r be positive real numbers and let $\alpha = \text{Prob.}(\nu\|Ax\|^{\sim} \leq \tau)$ where $\|Ax\|^{\sim}$ is defined with respect to the normal distribution on H. Let $\beta = 2/((2\pi)^{1/2}\nu r\alpha)$. If $\beta < 1$ then

$$\mu(S_r) \geq \frac{\left|\frac{1}{\alpha}E(\phi(\nu x)^{\sim}\chi(\|Ax\|^{\sim}))\right| - \beta}{1-\beta} \tag{2}$$

where $S_r = \{x : \|x\| \leq r\}$, $\chi(s)$ is the characteristic function of the interval $[0, \tau/\nu]$ and E denotes expectation with respect to the normal distribution.

PROOF. Let P_k be an increasing sequence of finite dimensional projections on H converging strongly to the identity operator. The distribution function of $\| Ax \|^{\sim 2}$ is continuous since $\| Ax \|^{2\sim}$ is a (possibly infinite) sum of independent random variables with continuous distribution functions and as is known convolution with a continuous distribution function yields a continuous function. Hence if $\alpha_k = \text{Prob.}(\| AP_k \|^{\sim} \leq \tau/\nu)$ then $\alpha_k \to \alpha$ as $k \to \infty$ since $\| AP_k x \|^{\sim}$ converges to $\| Ax \|^{\sim}$ in probability. Furthermore $\chi(\| AP_k x \|^{\sim})$ converges in probability to $\chi(\| Ax \|^{\sim})$ as $k \to \infty$. To see this we note that the random variables $\chi(\| AP_k x \|^{\sim})$ converge in measure to $\chi(\| Ax \|^{\sim})$ on the set where $| \; \| A(\cdot) \|^{\sim} - \tau/\nu | > \delta$ and since this set has arbitrarily small measure when δ is chosen small (since $\| Ax \|^{\sim}$ has a continuous distribution function) the assertion of the preceding sentence follows. Now consider the probability measure μ_k defined on $P_k H$ by means of $\mu_k(B) = \mu(P_k^{-1} B)$ for each Borel set B in $P_k H$. Its characteristic function is $\int_{P_k H} \exp(i(x,y)) \, d\mu \circ P_k^{-1}(x) =$

$\int_H \exp(i(P_k x,y)) \, d\mu(x) = \phi(y)$ for y in $P_k H$. If $\beta_k = 2((2\pi)^{1/2} \nu r \alpha_k)^{-1}$

then $\beta_k \to \beta$ so that for sufficiently large k we have $\beta_k < 1$. Hence by Lemma 2.1 we have

$$\mu_k(S_r^{(k)}) \geq \frac{|\alpha_k^{-1} \, E(\phi(\nu P_k y)^{\sim} \chi(\| AP_k y \|^{\sim}))| - \beta_k}{1 - \beta_k}$$

where $S_r^{(k)}$ denotes the central closed sphere of radius r in $P_k H$. The right side of this inequality converges to the right side of (2) as $k \to \infty$. Furthermore $\mu_k(S_r^{(k)}) = \mu(P_k^{-1} S_r)$. The sequence $P_k^{-1} S_r$ is a decreasing sequence of closed cylinders whose intersection is S_r for if x is in $\bigcap_{k=1}^{\infty} P_k^{-1} S_r$ then $P_k x$ is in S_r for all k. But as $x = \lim_{k\to\infty} P_k x$ and S_r is closed x is in S_r. It follows that $\lim_{k\to\infty} \mu(P_k^{-1} S_r) = \mu(S_r)$ and this establishes the lemma.

REMARK. We note that the quantity α in the previous lemma is strictly positive. This is established in [6, Lemma 1.1.1].

LEMMA 2.3. If $\{\mu_k\}$ is a sequence of probability measures on H with respective characterisitc functionals ϕ_k and if $\phi_k^{\sim}$ converges in probability to $\phi^{\sim}$ where ϕ is a uniformly $\mathcal{T}$ continuous function on H and $\phi(0) = 1$ then for every $\epsilon > 0$ there exists a sphere S_r in H such that $\mu_k(S_r) \geq 1 - \epsilon$ for all k.

PROOF. Let A be a Hilbert-Schmidt operator such that $|1 - \phi(x)| \leq \epsilon/4$ whenever $\| Ax \| \leq 1$. Let $\alpha = \text{Prob}\ (\| Ax \|^{\sim} \leq 1)$ and let r be so large that $\beta = 2/((2\pi)^{1/2}\ r\alpha) \leq 1/2$. If χ is the characteristic function of [0,1] then by Lemma 2.2 we have

$$\mu_k(S_r) \geq \frac{\alpha^{-1}|E(\phi_k(x)^{\sim}\chi(\| Ax \|^{\sim}))| - \beta}{1 - \beta}$$

Now insert $\phi_k(x)^{\sim} = 1 - (\phi(x)^{\sim} - \phi_k(x)^{\sim}) - (1 - \phi(x)^{\sim})$ into the right side of the last inequality and note that $E(\chi(\| Ax \|^{\sim})) = \alpha$. Furthermore $|1 - \phi(x)^{\sim}|\ \chi(\| Ax \|^{\sim}) \leq (\epsilon/4)\ \chi(\| Ax \|^{\sim})$ almost everywhere as follows from the fact that $\chi(\| Ax \|^{\sim}) = \text{lim. in prob. } \chi(\| AP_j x \|^{\sim})$ when P_j approaches the identity strongly through finite dimensional projections while $|1 - \phi(x)|\ \chi(\| Ax \|) \leq (\epsilon/4)\ \chi(\| Ax \|)$ for all x. Thus we obtain

$$\mu_k(S_r) \geq 1 - (1-\beta)^{-1}\ \alpha^{-1}\ E(|\phi(x)^{\sim} - \phi_k(x)^{\sim}|\ \ \chi(\| Ax \|^{\sim})) - 2\ \epsilon/4.$$

Since $\phi_k^{\sim}$ converges to $\phi^{\sim}$ in probability and all are bounded the second term of the last inequality approaches zero as $k \to \infty$. Hence for all sufficiently large k we have $\mu_k(S_r) \geq 1 - \epsilon$. By enlarging r we may obtain this inequality for all k.

LEMMA 2.4. If $\{\mu_k\}$ is a sequence of probability measures on a real separable Hilbert space H with characteristic functionals ϕ_k and if $\phi_k^{\sim}$ converges in probability to $\phi^{\sim}$ where ϕ is a uniformly $\mathcal{T}$ continuous function on H with $\phi(0) = 1$ then for any cofinite dimensional projection P on H $(\phi_k \circ P)^\sim$ converges in probability to $(\phi \circ P)^\sim$.

PROOF. We note first that for any projection P $\phi_k \circ P$ and $\phi \circ P$ are uniformly $\mathcal{T}$ continuous so that $(\phi_k \circ P)^\sim$ and $(\phi \circ P)^\sim$ exist. By assumption the projection $Q = I - P$ is finite dimensional. Regarding momentarily $\phi_k \circ P$ as a function on PH we see that there exists a finite dimensional projection P_k on H such that $P_k \leq P$ and such that Prob.$(|(\phi \circ P)^\sim - (\phi_k \circ R)^\sim| \geq 1/k) \leq 1/k$ whenever R is a finite dimensional projection with $P_k \leq R \leq P$. The sequence $\{P_k\}$ may and will be assumed increasing and have strong limit P. Since any finite dimensional projection on H may be dominated by a projection of the form S + Q where S is a finite dimensiona projection dominated by P we may also assume by enlarging the P_k if necessary that Prob.$(|\phi_k^\sim - \phi_k(S + Q)^\sim| \geq 1/k) \leq 1/k$ whenever S is a finite dimensional projection and $P_k \leq S \leq P$. In particular we have

$$\text{Prob.}(|(\phi_k \circ P)^\sim - (\phi_k \circ P_k)^\sim| \geq 1/k) \leq 1/k \tag{3}$$

and

$$\text{Prob.}(|\phi_k^\sim - (\phi_k \circ (P_k+Q))^\sim| \geq 1/k) \leq 1/k \tag{4}$$

for all k. Since $P_k + Q$ converges strongly to the identity $(\phi \circ (P_k+Q))^\sim$ converges in probability to $\phi^\sim$ by Corollary 5.3 of [7]. Furthermore from (4) and from the inequality $|\phi \circ (P_k + Q)^\sim - \phi_k \circ (P_k + Q)^\sim| \leq |\phi \circ (P_k + Q)^\sim - \phi^\sim| + |\phi^\sim - \phi_k^\sim| + |\phi_k^\sim - \phi_k \circ (P_k + Q)^\sim|$ it follows that $(\phi \circ (P_k + Q))^\sim - (\phi_k \circ (P_k + Q))^\sim$ converges to zero in probability.

We assert that $\phi_k(x)$ is strongly continuous in x uniformly in x and k. This follows from Lemma 2.3 for if $\epsilon > 0$ let r be so large that $\mu_k(S_r) \geq 1 - \epsilon$ for all k. If $|t| < \alpha$ implies $|e^{it}-1| < \epsilon$ then