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Petr N. Vabishchevich

ADDITIVE OPERATOR-DIFFERENCE SCHEMES

SPLITTING SCHEMES

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Preface

Applied mathematical modeling is basically concerned with the necessity to solve unsteady problems of mathematical physics. A mathematical model for simulation may include as elements both initial value problems for systems of ODEs and, which is most often, time-dependent PDEs. To construct discretization in space, finite difference schemes or finite element procedures are widely used in various ways. This results in transient problems for systems of ODEs. A specific feature of these problems of mathematical physics is in their high stiffness.

In this book, we study mathematical modeling problems in the corresponding finite-dimensional real Hilbert or Banach spaces as problems with the initial conditions for operator-differential equations. We investigate linear problems that are written in the form of evolutionary equations of first or second order and their systems. As a rule, these mathematical models are essentially nonlinear – the world is nonlinear, and, as academician Samarskii said, linear models comprise only a particular and very simple case. The primary linear models provide the basis for developing efficient computational algorithms, i.e., for designing elegant theoretical constructions that are used to verify their well-posedness and accuracy. Numerical methods for solving linear problems give us a methodological basis to construct algorithms for nonlinear problems.

Discretization in time is conducted using one or another difference approximation. This allows us to move from the Cauchy problem for operator-differential equations to operator-difference schemes. Unconditionally stable schemes are constructed employing implicit schemes. In view of stiffness of ODEs, explicit schemes have no practical interest. Optimization of computational algorithms for solving unsteady problems is associated with simplifications of the problem at the upper time level.

A typical situation is the case where the operators of the problem under consideration are represented as the sum of operator terms. Additive operator-difference schemes are attributed to the transition from a complex problem to a chain of simpler problems for the individual terms in this operator splitting. This splitting may have a different nature: the individual operators, e.g., may be associated with splitting with respect to the spatial variables or the decomposed parts may have different treatments in the sense of physical phenomena.

The classical examples of additive difference schemes are the ADI algorithms as well as locally one-dimensional schemes. They have been widely used in computational practice for more than half a century. Their study is based on the fundamental concept of summarized approximation. Nowadays, new classes of additive difference schemes are being developed. A major contribution to this research area is provided by the Russian (Soviet) school of computational mathematics.

The key results obtained in the theory and practical usage of splitting schemes are presented in detail in the book by Marchuk *Methods of Splitting*, 1989 (in Russian). In 1990 this book was published in English (Handbook of Numerical Analysis, Vol.1. Splitting and Alternating Direction Methods, Elsevier). New research results on the theory of additive schemes (schemes of splitting) are reflected in our joint book *Additive Schemes for Problems of Mathematical Physics* written with Samarskii. This book was published by Nauka, Moscow in 1999, in Russian, with a small edition. Unfortunately, it has actually gone unnoticed by English-speaking readers. This fact as well as the necessity to reflect the recent progress in constructing and studying additive schemes became the main reason for writing the new book.

The book is fundamentally concerned with constructing additive difference schemes to solve numerically unsteady multi-dimensional problems for PDEs. Two classes of schemes are highlighted: methods of splitting with respect to spatial variables (alternating direction methods) and schemes of splitting into physical processes. Regionally additive schemes (domain decomposition methods) are also developed for parallel computing. Unconditionally stable additive schemes of multicomponent splitting are considered for evolutionary equations of first and second order as well as for systems of equations. The matter of the book is primarily based on the results derived by the author and his co-authors during the last twenty years.

To present the material, we use the minimal mathematical tools concerned with the basic properties of operators in finite-dimensional spaces. The study of additive schemes is based on the general theory of stability for operator-difference schemes developed by Samarskii in the framework of finite-dimensional Hilbert spaces.

Moscow,
April, 2013

Petr N. Vabishchevich

Notation

A, B, C, D, S	difference operators
E	identity operator
A^*	adjoint operator
A^{-1}	inverse of the operator A
$A > 0$	positive operator ($(Ay, y) > 0$, if $y \neq 0$)
$A \geq 0$	non-negative operator ($(Ay, y) \geq 0$)
$A \geq \delta E,\ \delta > 0$	positive definite operator
$A_0 = \frac{1}{2}(A + A^*)$	self-adjoint part of the operator A
$A_1 = \frac{1}{2}(A - A^*)$	skew-symmetric part of the operator A
$A = \sum_{\alpha=1}^{p} A_\alpha$	p-componentwise splitting of the operator A
H	finite-dimensional real Hilbert space
$(\cdot, \cdot)$	scalar product in H
$\Vert \cdot \Vert$	norm in H
$(y, v)_A = (Ay, v)$	scalar product in H_A (operator $A = A^* > 0$)
$\Vert \cdot \Vert_A$	norm in H_A
$L_m(\omega)$	Banach space of grid functions, $m = 1, 2, \infty$
$\Vert \cdot \Vert_m$	norm in L_m
$\Vert A \Vert, \Vert A \Vert_m$	norm of a difference operator A
$\mu[A], \mu_m[A]$	logarithmic norm of a difference operator A
M, M_α	positive constants
Ω	computational domain
$\partial\Omega$	boundary Ω
∇, grad	gradient

div, $\operatorname{div}\mathbf{v} = \nabla \cdot \mathbf{v}$	divergence
ω	set of interior grid points
$\partial\omega$	set of boundary points
h, h_α	steps of a spatial grid
τ	time step
σ, σ_α	weights of a difference scheme
$y_x = \dfrac{y(x+h) - y(x)}{h}$	forward difference derivative at the point x
$y_{\bar{x}} = \dfrac{y(x) - y(x-h)}{h}$	backward difference derivative at the point x
$y_{\overset{\circ}{x}} = \dfrac{1}{2}(y_x + y_{\bar{x}})$	central difference derivative at the point x
$y_{\bar{x}x} = \dfrac{y_x - y_{\bar{x}}}{h}$	second difference derivative at the point x
$y = y^n = y(x, t^n)$	value of a grid function at the point x at the time level $t^n = n\tau$, $n = 0, 1, \ldots$

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