BERLINGHOFF

MATHEMATICS

THE

ART

OF

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MATHEMATICS: THE ART OF REASON

TO ROBIE

PREFACE

It is becoming fashionable, and indeed necessary, for all educated people to learn some mathematics. Within the past decade this attitude has been fostered by the increasing emphasis in the elementary and secondary school "new math" courses on reasoning, both inductive and deductive. This book is intended to provide a first course in college mathematics from the same point of view. It grew out of an effort to design a comprehensive, up-to-date, introductory mathematics course for college freshmen, especially those who do not intend to major in science or mathematics. In particular, the objectives of the book are:

1. To provide an introduction to the nature of mathematics as a major field of intellectual endeavor that is at least as much an art as it is a science;
2. To provide an acquaintance with the history of mathematics, so that the human aspect of the subject is not neglected;
3. To develop in the student abstract and rigorous thought processes; and
4. To foster an understanding of and facility with some of the basic concepts of contemporary mathematics.

Starting with some elementary logic and the concepts of *set* and *element*, the book develops much of the basic theory of sets, functions, and algebraic structures, motivating each step of the development by a consideration of one of the following two questions: Given any collection of things, how can we compare them and how can we combine them? Once the fundamental algebraic machinery has been established in Chapters II–IV, it is applied to the major topics of elementary mathematics in Chapters V–VIII. In Chapter V the various number systems are constructed, with brief excursions for discussions of numeration systems and number theory. Chapter VI begins with a consideration of the geometry of incidence; and then Euclidean geometry, the non-Euclidean geometries, and projective geometry are treated from both the axiomatic and algebraic viewpoints. Chapter VI culminates with a brief look at topology as a generalization of geometry. Chapter VII unites many of the concepts presented in the previous two chapters by relating geometry and algebra through a consideration of coordinate systems. Chapter VIII begins with a set-theoretic treatment of probability and then relates the theory to statistics. In Chapter IX the book turns to a more abstract question, investigating various kinds of infinity. After describing mathematical induction and the idea of limit, the comparison of types of infinity leads to transfinite arithmetic. The chapter ends with a brief discussion of the three major philosophies of mathematics, comparing their points of view especially with regard to the set theory just treated. Besides the historical comments within the chapters, there is an appendix that outlines the history of mathematics from prehistoric times to the present.

Although there are no specific prerequisites, it is expected that the reader's background includes at least two or three years of high school mathematics. Of course, students with better backgrounds will be able to progress more rapidly, and so the pace at which the material is covered will necessarily depend on the students involved. In general, there is sufficient material for a two-semester course that meets three times a week. If the students are mathematics majors, the depth of the course may be increased by the inclusion of material related to the discussion topics that are at the end of many sections and by an insistence on strict rigor in proofs of exercises. In this way the book may be used to provide freshman or sophomore mathematics majors with an introduction to abstract algebraic techniques and a "preview of things to come."

The contents of this book may be divided into three major parts relative to their use within a course. Chapter I and the historical Appendix are intended to provoke discussion about the nature of mathematics and to put the subject in historical perspective. Although they may be used anywhere in the course, they are especially useful as reading assignments at the beginning so that they may be referred to throughout the course. Chapters II, III, and IV should be covered in their entirety in the given order. (The only exception to this is Section 4 of Chapter IV, which may be omitted except for the definitions of ring, integral domain, and field.) The remaining chapters apply this machinery to various topics; therefore there is more flexibility in their use. In particular, Chapters VIII and IX are independent of each other, and either or both may be omitted if a shorter course is desired. Many important ideas are contained in the exercises. Hence, most of them should be done, or at least attempted.

There are many people to whom I owe a debt of gratitude for their encouragement and assistance in the preparation of this book. I am extremely grateful to Professor Dan E. Christie of Bowdoin College, who read the entire manuscript and made many valuable suggestions. I would also like to thank my colleagues at the College of Saint Rose, especially Sr. Kathleen Ann Bellcourt and Sr. Noel Marie Cronin, for their many helpful comments and inexhaustible patience while they taught preliminary versions of the text. Finally, I would like to thank Dr. Richard T. Wareham and Miss Martha W. Allen of D. C. Heath and Company for their kind assistance, and my wife Roberta for the many long hours she spent typing the manuscript.

William P. Berlinghoff

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I

ON THE NATURE OF MATHEMATICS

Mathematics is rooted historically in the empiricism of the Babylonians and other pre-Hellenic peoples, whose trial-and-error methods of boundary demarcation and building were eventually organized into numerical rules of procedure. Hence, from its earliest stages it has been a servant of man aiding his endeavors in other fields of human activity as a single exact language applicable to all fields that require reasoning. Today the general theories and procedures of mathematics simplify and refine the treatment of countless topics, and by translation into this common tongue the methods of many different disciplines are brought to bear on each other's problems. Our most common mathematical experience is the use of arithmetic in day-to-day business transactions, but one need not look very far to find mathematics serving man in a host of other fields. The physical scientists depend upon calculus and methods of analysis; the business world and the biological and social scientists are becoming involved more and more with statistical methods, probability, linear programming, and decision theory; philosophers recognize the importance of Boolean algebra as a tool for the study of logic; even the artist uses geometric ideas such as symmetry and projection to aid in his creative expression. These examples portraying the varied uses of mathematics in other fields could be multiplied many times over, but instead let us proceed to another viewpoint.

A science is characterized by its devotion to the discovery and organization of general truths and its concern for the operation and application of general laws. In this sense we shall claim that mathematics is a science, for the very heart of the subject is the establishment of orderly procedures and the study of the logical implications of statements. Even in your own mathematical experiences you can observe that Euclidean geometry possesses the characteristics of a science apart from any application it may have to the physical world, and the same may be said of algebra or any other branch of the field. Moreover, mathematics governs the employment of scientific principles and thus becomes the organizer of all science.

This characterization of mathematics is, however, far from sufficient. It is not merely a science. It is not even just the "queen of sciences," as some have said. There is in this field of thought a type of creativity found only in the fine arts. The men who first constructed the various systems of non-Euclidean geometry participated in a creativity quite similar to that of Rembrandt or Michelangelo. Mathematical imagination at least equals and often surpasses that required by the other fine arts, since the mathematician is neither aided nor confined by material forms of expression. His is a world of pure abstraction; he "lives in 'the wildness of logic' where reason is the handmaiden and not the master."† Although many mathematical theories have arisen in response to the challenge of specific problems in the natural or behavioral sciences, the creative mathematician often generalizes the original solution of a question and from it builds himself a logical edifice, posing and investigating questions of abstract structure without any regard to their connection with the world around him. He designs and constructs with a taste for order and harmony, pattern and symmetry, precision and generality. Nevertheless, some writers assert that the mathematician does not create any more than Leverrier and Adams created the planet Neptune or Admiral Peary created the North Pole. They claim that, since he is bound by reason, the man who designs a mathematical system is merely discovering one of the many patterns of thought that already exist as logical consequences of the various initial statements used as hypotheses. One could reply that this is tantamount to asserting that Rodin discovered "The Thinker" since he merely shaped a piece of stone into one of the many forms it was already capable of assuming; however, the roots of this question lie deep in philosophy and we shall not attempt to resolve the controversy here.

It should be apparent by now that the nature of mathematics defies simple description. As an art mathematics creates new worlds, and as a science it explores them. It is a common unifying force present in all human intellectual endeavor, forever broadening the horizons of the mind, exploring virgin territory, and organizing new information into weapons for another assault on the unknown. "It is a language, a tool, and a game—a method of describing things conveniently and efficiently, a shorthand adapted to playing the game of common sense."‡ It is "the subject in which we never know what we are talking about nor whether what we say is true,"§ and yet it is basic to the organization and interpretation of all truth. It demands a novelist's imagination, a poet's perception of analogy, an artist's appreciation of beauty, and a politician's flexibility of thought. Mathematics is indeed an integral and indispensable part of every truly liberal education. It is "the thinking man's liberal art."||

FOR DISCUSSION

Compare mathematics with the various fine arts that you know, indicating similarities and differences.

† [41], p. 612.
‡ [53], p. 90.
§ Bertrand Russell, *International Monthly*, Vol. 4, 1901.
|| [29], p. 113.

II

THE FOUNDATIONS OF MATHEMATICS

1. LOGIC

If one were required to single out the dominant characteristic of mathematics, it would have to be "reasonableness." In fact, mathematics is utterly dependent upon man's innate rationality, and hence if we are to discuss mathematics at all, we must begin with a consideration of that elusive concept we call reason. But since every intellectual consideration is by nature rational, we are forced to use reason to analyze reason, an awkward situation at best, leading to intricacies whose investigation we leave to courses in philosophy. We must content ourselves with a brief exposition of those basic logical principles that are essential to mathematical discourse.

The notion underlying all discourse is that of a meaningful statement. Such a statement is called a **proposition**, and each proposition has a **truth value**, which in our logical system is either **true** or **false**, depending upon the context in which the statement appears. The assertion that there is no category between *true* and *false* in which we may place a statement is usually called the Law of the Excluded Middle. We also assert that a statement may not simultaneously possess both truth values. This is the Law of Contradiction.

From any proposition we may derive several related propositions:

(a) the **negation** or **contradictory** of the proposition, a statement characterized by the fact that it always has the opposite truth value from the original proposition, and

(b) **contrary** propositions, statements that are false whenever the original proposition is true, but may still be false when the original proposition is false.

EXAMPLE: Consider the proposition "The wall is red." Its negation is "The wall is not red." Several contrary propositions are "The wall is green," "The wall is yellow," and "The wall is blue." The proposition "The wall is concrete" is neither the negation nor a contrary of the original proposition.

NOTATION: Propositions are usually denoted by small **italic** letters, especially p and q. The negation of a proposition p is denoted by $\sim p$.

Propositions may also be categorized with regard to their scope, and thus are either universal or existential. A **universal proposition** is an assertion about all things of a certain kind, whereas an **existential proposition** merely asserts the existence of at least one thing that satisfies the statement. The negation of a universal proposition is existential, and the negation of an existential proposition is universal.

EXAMPLES: "All buildings have flat roofs" is a universal proposition; its negation is "There exists a building that does not have a flat roof."

"There are pink elephants" is an existential proposition; its negation is "No elephants are pink." (i.e., "All elephants possess the property of not being pink.")

NOTATION: $\forall$ means "(for) all"; $\exists$ means "there exists." Thus, the universal statement "All buildings have flat roofs" may be written formally as "B has a flat roof, $\forall$ buildings B," and the existential statement "There are pink elephants" may be written "$\exists$ a pink elephant."

Closely allied to the notion of proposition is that of **propositional function,** an expression in the form of a statement but without truth value that simply states a relationship involving symbols or variables whose meanings have not been determined.

EXAMPLES: "Every A contains a B." "There is a gznk in the parking lot."

Now that we have considered propositions singly, we move to a consideration of relationships that exist between pairs of propositions. The most basic of these is *implication*. A strict definition could be given for "implies,"† but for our purposes it will suffice to say that "p **implies** q" means that the truth of proposition p insures the truth of proposition q; that is, if p is true, then q is true. We say that p is a **sufficient** condition for q, and q is a **necessary** condition for p. Notice that if p and q are specific propositions, the statement "p implies q" is itself a proposition, called a **conditional** or an **implication**, whose truth value is "false" if and only if p is true and q is false. In all other cases, the conditional is considered a true statement. p is called the **hypothesis** and q the **conclusion.** If we take any conditional and interchange its hypothesis and conclusion, we obtain the **converse** of that conditional.

EXAMPLES: "If it is spring, then the grass is green" is a conditional in which "It is spring" is the hypothesis and "The grass is green" is the

† Strictly, "p implies q" means "p is false or q is true."

conclusion. The converse is "If the grass is green, then it is spring."

NOTATION: "p implies q" is written $p \Rightarrow q$, and its converse is $q \Rightarrow p$.

The truth value of a conditional is not related to that of its converse; that is, a true conditional may or may not have a true converse. If it does, then the hypothesis and conclusion are said to be **equivalent**. The two most common ways of expressing the equivalence of propositions p and q is by saying "p is necessary and sufficient for q" or "p is true if and only if q is true." This type of statement is called a **biconditional** or an **equivalence**.

EXAMPLE: "Two sides and the included angle of triangle T_1 are congruent respectively to two sides and the included angle of triangle T_2" and "Triangles T_1 and T_2 are congruent" are equivalent propositions.

NOTATION: "If and only if" is abbreviated as *iff* or symbolized by $\Leftrightarrow$.

The truth of an implication is usually referred to as **validity**; that is, an implication is said to be valid if its conclusion follows from its hypothesis. The fact that an implication is valid does not insure the truth of its conclusion; for this we also need the truth of the hypothesis. A **deductive argument** is simply a concatenation of implications, in which the conclusion of each implication is at least part of the hypothesis of the next, and the argument is valid if and only if each implication is valid. The truth of its conclusion, however, also requires the truth of the initial hypothesis. Thus, a deductive argument is simply a process for guaranteeing that the truth of a certain statement (conclusion) follows from the truth of one or more other statements (hypotheses). The establishment of the truth of a proposition by making it the conclusion of a deductive argument whose initial hypothesis is taken as true is called a **direct proof** of the proposition.

There is another type of proof based on both deduction and the negation of a proposition. Since a true hypothesis and a valid argument together must yield a true conclusion, then a valid argument that yields a false conclusion must proceed from a false hypothesis. Hence we may also prove the truth of a proposition p by forming its negation, $\sim p$, and using $\sim p$ as the hypothesis of a valid argument whose conclusion is false. This implies $\sim p$ must be false, and hence p must be true by the way we defined negation. This procedure is known as **proof by contradiction** or **indirect proof**. The direct and indirect methods of proof are exemplified by the deductive arguments used throughout the remainder of this book.

EXERCISES

1. Give the negation and a contrary of each of the following propositions:
   (a) The flower is purple.
   (b) All freshmen are flowers.

(c) Some birds are fire engines.
(d) The development of the number systems is motivated exclusively by a consideration of algebraic equations.
(e) Several meetings have been held to discuss the renovation of the schedule.
(f) If rabbits eat carrots, then they have good eyesight.
(g) If an algebraic expression is a quadratic equation, then it generates a conic section.

2. In each of the following conditionals, state the hypothesis, the conclusion, and the converse:
(a) If all flying things are airplanes, then birds are airplanes.
(b) p is true if q is true.
(c) The existence of a propositional function implies the existence of an infinity of propositions.
(d) Truth implies truth.
(e) We will be able to sleep through this lecture if nobody asks a question.
(f) If there are sufficient funds and if the carpenters do not strike, the entire building will be renovated by September.
(g) Every chipmunk is a fish.
(h) No snowmen are purple.

3. Let the propositions p and q be "Roses are red" and "Snowmen like carrots," respectively. Translate into acceptable English:
(a) $p \Rightarrow q$ **(b)** $q \Rightarrow p$ **(c)** $p \Leftrightarrow q$
(d) $(\sim p) \Rightarrow q$ **(e)** $p \Rightarrow (\sim q)$ **(f)** $(\sim q) \Rightarrow (\sim p)$

4. Follow the directions of Exercise 3 for the propositions
p: "Several errors have been made," and
q: "All the answers are incorrect."

2. THE AXIOMATIC METHOD

Any body of knowledge is organized and transmitted basically by means of the principles outlined in the previous section, and mathematics is no exception. Deductive reasoning, however, only supplies the method of procedure; it gives no indication of where to start. The quest for this starting point leads naturally to an investigation of meanings of words. In order to arrive at a common understanding and remove all ambiguity from future discussion it is necessary to define the words we use. The concept of definition involves the statement of a characteristic property; that is, if we are to define a word, we must state a condition such that,

(1) given any object whatsoever, we can determine whether or not that object satisfies the condition, and
(2) the word being defined is attached to an object if and only if it satisfies that condition.

What we are doing, essentially, is developing a system of name tags for ideas. Thus, to "define" a word by merely giving a synonym is either meaningless or useless. If we do not know what the synonym means, we have no criterion by which to apply the name; if we do know what the synonym means, it is a perfectly good name for that idea and there is no need to confuse the issue by supplying another.

Of course, just as one cannot learn Arabic solely by using an all-Arabic dictionary, so it is not possible to define every word we use. Any attempt to do this would simply result in a circular set of statements, each dependent on another, and hence all meaningless. Therefore, we must begin any logical undertaking with one or more undefined terms. Similarly, not every statement can be proved from previous ones. We must have some initial hypotheses, statements assumed true without proof. These are usually called **axioms** or **postulates.**

Euclid recognized these principles and put them to use in his *Elements.* He regarded geometry as a description of the physical world and attempted to systematize this description by placing it on a deductive foundation. He defined all the technical terms used, but in doing so employed other words such as *part, length, equal,* etc., without defining them. He also distinguished between axioms and postulates, and in this respect was in agreement with the majority of early Greek scholars, although the precise nature of this distinction was open to dispute. Euclid's axioms were statements of common ideas that he regarded as obvious, such as "The whole is greater than any of its parts" and "If equals be subtracted from equals, the remainders are equal," whereas his postulates were statements dealing explicitly with geometry, such as "A straight line can be drawn from any point to any point" and "All right angles are equal." These latter statements he regarded as idealizations of physical truths and therefore immutable, in a sense. They were looked upon as true observations about the nature of the physical world rather than as arbitrary assumptions. The *Elements* typify what has come to be known as **material axiomatics.**

Contrasted with material axiomatics is the modern deductive approach, called **formal axiomatics**, based upon assumptions that are not considered to have any *a priori* truth value. This type of system begins with some **undefined terms**, words that have no meaning at all in the system and behave much the same as algebraic symbols. With these words some propositions (actually, propositional functions) are constructed, and although they have no truth value as such, they are assigned the value *true* for use within the system. These are the axioms or postulates. (The two words are interchangeable in formal axiomatics.) From there on, all new words used in the system are defined in terms of the undefined terms, and all subsequent statements are proved deductively from the axioms. The proved statements are called **theorems**. This type of structure is called an **abstract mathematical system**, and the totality of such systems constitutes **pure mathematics**.

Lest we be accused of saying that all mathematicians are either psychic or phenomenally lucky, let us clarify this approach a bit. Mathematicians do not choose arbitrary symbols and statements at random and just happen to come upon useful systems most of the time; they often have in mind a concrete system that they are attempting to describe, just as Euclid did. The essential difference is that they recognize that the concrete system they are describing does not predetermine the abstract system they are formulating. The postulates are not propositions that are true by nature, but propositional functions that have no intrinsic truth value and are considered as true in the system merely by agreement. The abstract system is, in general, subject to many interpretations of which its prototype is only one. An interpretation is called a **model,** and is formed by assigning meanings to the undefined terms and then verifying the truth of the postulates with respect to these meanings. **Applied mathematics** is the study of models of pure mathematical systems.

EXAMPLE: Let us consider the following system:

Undefined terms: *bird, cage, belong to.*

Axioms: 1. There exist at least one bird and at least one cage.
2. If X and Y are distinct birds, then there is at least one cage that belongs to both of them.
3. If X and Y are distinct birds, then there is at most one cage that belongs to both of them.
4. If x and y are distinct cages, then there is at least one bird that belongs to both of them.
5. At least three birds belong to any cage.
6. Not all birds belong to the same cage.

This is an abstract mathematical system which we shall call the axiom system S.

Now let us examine a model of S. Let A, B, C, D, E, F, and G be brass rings connected by wires 1, 2, 3, 4, 5, 6, and 7 in the following manner:

Wire 1 connects rings A, B, and C.
Wire 2 connects rings A, D, and E.
Wire 3 connects rings A, F, and G.
Wire 4 connects rings B, D, and F.
Wire 5 connects rings B, E, and G.
Wire 6 connects rings C, D, and G.
Wire 7 connects rings C, E, and F.
(See Figure 1.)

If we interpret *bird, cage*, and *belong to* as *brass ring, wire*, and *be attached to*, respectively, it is fairly easy to see that all the axioms of S are satisfied, and hence this is a physical model of S.

A final remark regarding terminology is in order. The undefined terms of our example S were deliberately chosen for their obscurity to emphasize the fact that they have no meaning other than the significance imparted to them by the