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Conformal Dynamics and Hyperbolic Geometry

Conference on Conformal Dynamics
and Hyperbolic Geometry
in Honor of Linda Keen's 70th Birthday
Graduate School and University Center of CUNY
New York, NY
October 21–23, 2010

Francis Bonahon
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Conformal Dynamics and Hyperbolic Geometry

Preface

This book is a collection of papers based on activity at the *Conference on Conformal Dynamics and Hyperbolic Geometry* held on October 21st to 23rd, 2010, in celebration of Linda Keen's seventieth birthday and sponsored by Lehman College, the Graduate Center of CUNY and the National Science Foundation.[1] The articles presented here fit into a grand strategy, which is to develop mathematical techniques that provide a foundation for understanding one dimensional real and complex dynamics. The topics include iteration of rational and holomorphic maps, the geometry of Fuchsian and Kleinian groups and objects that in the limit have asymptotically conformal structure including the universal hyperbolic solenoid and smooth circle expanding maps. Some of the articles go directly to the fractal and chaotic nature of the dynamical phenomena so richly displayed in many of the diagrams given herein and others focus primarily on tools and types of arguments that come mainly from complex analysis, hyperbolic geometry and Teichmüller theory.

This book will be useful for beginners and a primary source for young mathematicians looking for interesting research problems. It is therefore a fitting tribute to Professor Keen, who has done so much to make our CUNY Mathematics Ph.D. Program a hub of research for students and faculty alike and to support the significant number of mathematicians around the world who study these topics.

[1]The conference acknowledges support from three sources: the Graduate Center of CUNY, Lehman College and the National Science Foundation Grant DMS 1042777.

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Normal families and holomorphic motions over infinite dimensional parameter spaces

Michael Beck, Yunping Jiang, and Sudeb Mitra

ABSTRACT. We use Earle's generalization of Montel's theorem to obtain some results on holomorphic motions over infinite dimensional parameter spaces. We also study some properties of group-equivariant extensions of holomorphic motions.

1. Introduction

The main goal in this paper is to study an application of Earle's generalization of Montel's theorem ([**3**]) to holomorphic motions over infinite dimensional parameter spaces. For precise definitions see §1.1. In the study of holomorphic motions, an important question is the following: given a holomorphic motion $\phi : V \times E \to \widehat{\mathbb{C}}$, where E is a finite set consisting of n points, if $a \in \widehat{\mathbb{C}} \setminus E$, does there exist a holomorphic motion $\widehat{\phi} : V \times (E \cup \{a\}) \to \widehat{\mathbb{C}}$ such that $\widehat{\phi}$ extends ϕ? In their famous paper [**9**], Sullivan and Thurston called this the "holomorphic axiom of choice." If $\phi : \Delta \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion, where Δ is the open unit disk in the complex plane, and E is any subset of $\widehat{\mathbb{C}}$, the holomorphic axiom of choice is the crucial step in extending ϕ to a holomorphic motion of $\widehat{\mathbb{C}}$; see, for example, [**2**] and [**9**]. In our paper, we use a theorem of Earle to generalize this fact to holomorphic motions over connected complex Banach manifolds. More precisely, we show that if V is a connected complex Banach manifold with a basepoint such that the holomorphic axiom of choice holds, then any holomorphic motion $\phi : V \times E \to \widehat{\mathbb{C}}$ can be extended to a holomorphic motion $\widehat{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Furthermore, if the holomorphic motion ϕ is group-equivariant, then the extended holomorphic motion $\widehat{\phi}$ can be chosen to have the same group-equivariance property.

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1.1. Definitions and some facts.

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DEFINITION 1.1. Let V be a connected complex manifold with a basepoint t_0 and let E be any subset of $\widehat{\mathbb{C}}$. A *holomorphic motion* of E over V is a map $\phi : V \times E \to \widehat{\mathbb{C}}$ that has the following three properties:

(i) $\phi(t_0, z) = z$ for all z in E,

(ii) the map $\phi(t, \cdot) : E \to \widehat{\mathbb{C}}$ is injective for each t in V, and

(iii) the map $\phi(\cdot, z) : V \to \widehat{\mathbb{C}}$ is holomorphic for each z in E.

We say that V is a *parameter space* of the holomorphic motion ϕ. We will assume that ϕ is a *normalized* holomorphic motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi(t, \cdot)$ for every t in V. It is sometimes useful to write $\phi(t, z)$ as $\phi_t(z)$, and also as $\phi^z(t)$, for $(t, z) \in V \times E$.

If E is a proper subset of $\widehat{E}$ and $\phi : V \times E \to \widehat{\mathbb{C}}$, $\widehat{\phi} : V \times \widehat{E} \to \widehat{\mathbb{C}}$ are two holomorphic motions, we say that $\widehat{\phi}$ *extends* ϕ if $\widehat{\phi}(t, z) = \phi(t, z)$ for all (t, z) in $V \times E$.

DEFINITION 1.2. Let V be a connected complex manifold with a basepoint. Let G be a group of Möbius transformations, let $E \subset \widehat{\mathbb{C}}$ be G-invariant, which means, $g(E) = E$ for each g in G. A holomorphic motion $\phi : V \times E \to \widehat{\mathbb{C}}$ is **G-equivariant** if for any $t \in V, g \in G$ there is a Möbius transformation, denoted by $\theta_t(g)$, such that

$$\phi(t, g(z)) = (\theta_t(g))(\phi(t, z))$$

for all z in E.

The following generalization of Montel's Theorem, due to Earle (see [**3**]), is important in our paper.

THEOREM 1.3. *Let V be any connected complex Banach manifold, let $\mathcal{F}$ be any family of holomorphic functions $f : V \to \mathbb{C}$ such that the range of f never contains 0 or 1. Then $\mathcal{F}$ is a normal family, meaning that if $\{f_\alpha\}$ is any net in $\mathcal{F}$, there is a subnet $\{f_\beta\}$ which converges in the compact-open topology.*

We now review a well-known fact. For holomorphic motions over Δ, this was proved in [**7**].

PROPOSITION 1.4. *Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion, where V is a connected complex Banach manifold with basepoint t_0. Let ρ_V denote the Kobayashi pseudometric on V. Then:*

(1) *$\phi(\cdot, \cdot)$ is jointly continuous.*

(2) *ϕ extends to a holomorphic motion to the closure $\overline{E}$.*

(3) *$\phi_t : E \to \widehat{\mathbb{C}}$ is the restriction of a (normalized) quasiconformal self-map of $\widehat{\mathbb{C}}$.*

PROOF. Let ρ be the Poincaré distance on $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Note that if $z, w \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ are a bounded hyperbolic distance apart, and $|z| \to 0$, then $|w| \to 0$. Define $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\eta(M, \epsilon) := \sup\{|w| : \rho(z, w) \le M, |z| \le \epsilon\}$. Evidently this function is continuous, increasing and unbounded in ϵ for each fixed M, and moreover $\eta(M, \epsilon) \to 0$ as $\epsilon \to 0$, and $|w| \le \eta(M, |z|)$ whenever $\rho(z, w) < M$.

For any four distinct points $a, b, c, d \in E$ define:

$$g(t) := cr(\phi_t(a), \phi_t(b), \phi_t(c), \phi_t(d)),$$

the cross-ratio of the points $\phi_t(a), \phi_t(b), \phi_t(c), \phi_t(d)$. So, we have

$$g(t) = \frac{(\phi_t(a) - \phi_t(c))(\phi_t(b) - \phi_t(d))}{(\phi_t(a) - \phi_t(d))(\phi_t(b) - \phi_t(c))}.$$

Since ϕ is injective in the second coordinate, this gives a mapping $g : V \to \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Since ϕ is holomorphic in the first coordinate, g is holomorphic and thus $\rho(g(t), g(u)) \leq \rho_V(t, u)$ for all $t, u \in V$. Since $g(t_0)$ is equal to $cr(a, b, c, d)$, we have:

$$|cr(\phi_t(a), \phi_t(b), \phi_t(c), \phi_t(d))| \leq \eta(\rho_V(t, t_0), |cr(a, b, c, d)|).$$

Keep b and d fixed, and let $a \to c$. Then $cr(a, b, c, d) \to 0$, whence it follows $\phi_t(a) \to \phi_t(c)$ uniformly with modulus of continuity depending only on $\rho_V(t, t_0)$. Since ϕ is continuous in the first coordinate this gives the first statement, that of joint continuity.

For (2), using the above arguments, for any fixed t, ϕ_t is equicontinuous on E, and therefore, it can be extended to a continuous function on $\overline{E}$. For any fixed $z \in \overline{E}$ ($z \neq 0, 1, \infty$), let $z_n \to z$, where $z_n \in E$. Since $\phi^{z_n}(t)$ is holomorphic for each z_n, and $z_n \neq 0, 1, \infty$ for any n, $\{\phi^{z_n}(t)\}$ is a normal family. Therefore, there exists a subsequence $\phi^{z_{n_i}} \to \phi^z$ and ϕ^z is holomorphic by Theorem 1.3. For the injectivity, since for any $z \neq w \in \overline{E}$, the cross-ratio $cr(0, \phi(t, z), \phi(t, w), \infty)$ is bounded, this implies $\phi(t, z) \neq \phi(t, w)$.

For (3) consider any point $z \in \overline{E}$, any other two points, $w_1, w_2 \in \overline{E}$ such that $cr(z, w_1, w_2, \infty) = 1$, then $cr(\phi_t(z), \phi_t(w_1), \phi_t(w_2), \infty) \leq \eta(\rho(t, t_0))$, this implies that ϕ_t is the restriction of a quasiconformal self-map of $\widehat{\mathbb{C}}$. □

REMARK 1.5. For standard facts on quasiconformal mappings see [**1**]. The extension to the closure (Part 2) is also proved in Theorem 1 in [**5**], using different methods.

DEFINITION 1.6. Let V be a connected complex Banach manifold with a basepoint. Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion of any finite set E (containing 0, 1, and ∞), such that if a is any point in $\widehat{\mathbb{C}} \setminus E$, there exists a holomorphic motion $\widehat{\phi} : V \times (E \cup \{a\}) \to \widehat{\mathbb{C}}$ extending ϕ. Then we say that the *holomorphic axiom of choice* holds.

1.2. Statements of the main theorems. Our goal in this paper is to prove the following theorems.

Theorem A. *Let V be any connected complex Banach manifold with a basepoint t_0 such that the holomorphic axiom of choice holds. Then, if E is any subset of $\widehat{\mathbb{C}}$, and if $\phi : V \times E \to \widehat{\mathbb{C}}$ is a holomorphic motion, ϕ can be extended to a holomorphic motion of $\widehat{\mathbb{C}}$.*

In the next theorem, E is a closed G-invariant subset of $\widehat{\mathbb{C}}$; see Definition 1.2.

Theorem B. *Let V be a connected complex Banach manifold with basepoint t_0, such that the holomorphic axiom of choice holds. Then, if $\phi : V \times E \to \widehat{\mathbb{C}}$ is a G-equivariant holomorphic motion, ϕ can be extended to a G-equivariant holomorphic motion of $\widehat{\mathbb{C}}$.*

2. Proof of Theorem A

LEMMA 2.1. *Let V be a connected complex Banach manifold with basepoint t_0, let E be a finite subset of $\widehat{\mathbb{C}}$. Let $\Phi := \{$all normalized holomorphic motions $\phi : V \times E \to \widehat{\mathbb{C}}$ with basepoint $t_0\}$. Then Φ is a compact set in the compact-open topology.*

PROOF. By Proposition 1.4, each element of Φ is a jointly continuous function; so speaking of Φ as a subset of the space of all continuous functions on $V \times E$ with the compact-open topology makes sense. To show Φ is compact, we must show every net in Φ has a subnet converging to a limit in Φ. Let $z_1, \ldots, z_n$ be the elements of $E \setminus \{0, 1, \infty\}$.

Let $\{\phi_\alpha\}$ be a net in Φ, and consider $\{\phi_\alpha^1\}(t) := \phi_\alpha(t, z_1)$. This defines a family of holomorphic functions on V which miss 0 and 1, so by Theorem 1.3 there is a convergent subnet $\{\phi_\beta^1\}$. Consider next the net $\{\phi_\beta^2\}$, with like notation. By the same result there is a subnet $\{\phi_\gamma^2\}$ which converges compactly, $\{\phi_\gamma^1\}$ converges compactly as well. Repeating this argument we obtain a net $\{\phi_\delta\}$ such that each $\{\phi_\delta^k\}$ converges compactly. Setting $\phi(t, z_k) := \lim_\delta \phi_\delta^k(t)$, and setting $\phi(t, \zeta) := \zeta$ if $\zeta = 0, 1, \infty$, defines a function $\phi : V \times E \to \widehat{\mathbb{C}}$. If we can show $\phi \in \Phi$, it will be the limit desired.

That $\phi(t_0, z) = z$ for all $z \in E$ is obvious. That ϕ is holomorphic in the first coordinate follows from the fact each ϕ_δ^k is holomorphic, and the collection of holomorphic functions is closed in the compact-open topology. Also, the limit function ϕ is evidently normalized. Showing ϕ is injective in the second coordinate is done as follows.

Fix $t \in V$. Since each $\phi_\delta(t, z) \in \Phi$, there exists, (by Proposition 1.4) an η, independent of δ, such that:

$$|cr(\phi_{\delta,t}(z), 1, 0, \phi_{\delta,t}(z'))| \leq \eta(|cr(z, 1, 0, z')|).$$

and with z and z' distinct elements of E not equal to 0 or 1. Passing to the limit gives:

$$|cr(\phi_t(z), 1, 0, \phi_t(z'))| \leq \eta(|cr(z, 1, 0, z')|).$$

The cross-ratio on the RHS will be $< \infty$, so the cross-ratio on the LHS will be $< \infty$, implying $\phi_t(z) \neq \phi_t(z')$, thus proving injectivity in the second coordinate in this case. The possibility z or z' is equal to 0 or 1 is dealt with by replacing 0 or 1 with ∞ and then permuting elements in the cross-ratios above. □

LEMMA 2.2. *Let $\{E_n\}$ be an ascending sequence of finite subsets of $\widehat{\mathbb{C}}$ such that $E_1 \supset \{0, 1, \infty\}$, and let $E = \overline{\bigcup_n E_n}$. For each n, let ϕ_n be a normalized holomorphic motion on $V \times E_n$, where as usual V is a complex connected Banach manifold with basepoint t_0. Then there is a subsequence ϕ_{n_j}, and a holomorphic motion $\phi : V \times E \to \widehat{\mathbb{C}}$, such that ϕ_{n_j} converges compactly to ϕ on each $V \times E_n$.*

PROOF. Denote $\bigcup_n E_n$ by E' for convenience. Since $\phi_n|(V \times E_1)$ is a collection of holomorphic motions of E_1, and E_1 is finite, by Lemma 2.1, there is a subsequence $\phi_{n_{k1}}$ which converges compactly on $V \times E_1$. Since $\phi_{n_{k1}}|(V \times E_2)$ is a sequence of holomorphic motions on $V \times E_2$ there is, by the same lemma, a further subsequence $\phi_{n_{k2}}$ which converges compactly on $V \times E_2$, and therefore on $V \times E_1$ as well. Continuing like this, and then applying a diagonalization argument, we see that there is a sequence $\phi_{n_{kj}}$ which converges compactly on each $V \times E_n$. Therefore, it converges to a limit $\phi : V \times E' \to \widehat{\mathbb{C}}$ which is a holomorphic motion. By Proposition 1.4, this extends to a holomorphic motion of $\overline{E'}$. □

Proof of Theorem A.

Step 1: By Proposition 1.4, we can assume that E is closed. Let $\{E_n\}$ be an ascending sequence of finite subsets of E whose union $E^{'}$ is dense in E, and let $y \in \widehat{\mathbb{C}} \setminus E$. We claim ϕ has an extension ϕ' on $V \times (E \cup \{y\})$ which is also a holomorphic motion.

Let ϕ_n be the holomorphic motion on $V \times E_n$ obtained by restricting ϕ, and let ϕ'_n be a holomorphic motion on $V \times (E_n \cup \{y\})$ which extends ϕ_n. By Lemma 2.2, there is a subsequence ϕ'_{n_j} which converges at each point of $E' \cup \{y\}$ to a holomorphic motion on $V \times (E' \cup \{y\})$. By Proposition 1.4, this holomorphic motion can be extended to a holomorphic motion on $E \cup \{y\}$; denote it by ϕ', and since it agrees with ϕ on the dense subset $V \times E'$ of $V \times E$, and since both are continuous, ϕ' is the extension desired.

Step 2: Let $E \subset \widehat{\mathbb{C}}$ be any closed set, and let $Y = \{y_1, y_2, \ldots\}$ be a countable dense subset of $\widehat{\mathbb{C}} \setminus E$. Let $F_0 = E$, let $F_1 = E \cup \{y_1\}$, let $F_2 = F_1 \cup \{y_2\}$, and so on. Let $\phi_0 = \phi$. By Step 1 there is an extension ϕ_1 to $V \times F_1$ of ϕ_0 which is also a holomorphic motion. By Step 1 again, there is an extension ϕ_2 to $V \times F_2$ of ϕ_1 which is also a holomorphic motion. Continuing inductively, we obtain a sequence $\phi_n : V \times F_n \to \widehat{\mathbb{C}}$ of holomorphic motions, all of which extend ϕ. Since each holomorphic motion is an extension of the one before, a holomorphic motion ϕ' clearly exists on $V \times (E \cup Y)$. Use Proposition 1.4, and we are done by choice of Y. □

PROPOSITION 2.3. *Let V be a connected complex Banach manifold with basepoint t_0. Let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a holomorphic motion with the following property: if E_0 is a finite subset of E, and $Y \subset \widehat{\mathbb{C}} \setminus E_0$ is finite, there is a holomorphic motion $\widetilde{\phi}$ on $V \times (E_0 \cup Y)$ whose restriction to $V \times E_0$ agrees with ϕ. Then there is a holomorphic motion $\widehat{\phi} : V \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which extends ϕ.*

PROOF. By Proposition 1.4 we may assume that E is a closed set. Let $\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \ldots...$ be an ascending sequence of finite subsets of E whose union $E^{'}$ is dense in E. Let $Y = \{y_1, y_2, ...\}$ be a countable dense subset of $\widehat{\mathbb{C}} \setminus E$, and let $F_n := E_n \cup \{y_1, y_2, \ldots, y_n\}$ for every n. By hypothesis there is for each n a holomorphic motion ϕ_n on $V \times F_n$ whose restriction to $V \times E_n$ coincides with ϕ. By Lemma 2.2, if $F' := \cup F_n$ there is a holomorphic motion ϕ' on $V \times F'$ such that ϕ' agrees with ϕ on $V \times E'$. Let $\widehat{\phi}$ be the extension of this motion to the closure of F', it will extend ϕ and since $\overline{F'} = \widehat{\mathbb{C}}$, this is the extension desired. □

3. Group-equivariant extensions of holomorphic motions

The discussion in Sections 3 and 4 are inspired by the arguments in the proof of Theorem 1 in [**4**]. Let V be a connected complex Banach manifold with basepoint t_0, let G be a group of Möbius transformations, and E be a closed G-invariant subset of $\widehat{\mathbb{C}}$ (containing 0, 1, ∞). Suppose $\phi : V \times E \to \widehat{\mathbb{C}}$ is a G-equivariant holomorphic motion (see Definition 1.2).

For any $t \in V, g \in G$, there is a Möbius transformation, denoted by $\theta_t(g)$, such that

$$\phi(t, g(z)) = (\theta_t(g))(\phi(t, z))$$

for all z in E. It is proved in Theorem 4 (i) of [**8**] that $\{\theta_t\}_{t \in V}$ is a holomorphic family of isomorphisms of G; see Definition 1.10 and Theorem 4 (i) of [**8**]. Since θ_{t_0} is the identity, θ_t is a quasiconformal deformation of G, for all t in V, by Theorem 4 (ii) of [**8**]; which means, there exists a quasiconformal homeomorphism f_t of $\widehat{\mathbb{C}}$ inducing θ_t in the sense that

$$f_t \circ g = \theta_t(g) \circ f_t \qquad \text{for all } g \in G.$$

In particular each of the isomorphisms θ_t is type-preserving.

If G is a group of Möbius transformations and $z \in \widehat{\mathbb{C}}$ then the stabilizer is denoted by G_z for the remainder of the paper.

PROPOSITION 3.1. *Let V be a connected complex Banach manifold with basepoint t_0, let G be a group of Möbius transformations, let E be a closed G-invariant subset of $\widehat{\mathbb{C}}$ containing $\{0, 1, \infty\}$, and let $\phi : V \times E \to \widehat{\mathbb{C}}$ be a G-equivariant holomorphic motion. Let $F = \{z \in \widehat{\mathbb{C}} : G_z \neq \{id\}\}$. Then ϕ has an extension $\widetilde{\phi} : V \times (E \cup F) \to \widehat{\mathbb{C}}$ which is also a G-equivariant holomorphic motion.*

Proof. Since E is closed and G-invariant and contains at least three points, it contains all fixed points of parabolic or loxodromic (including hyperbolic) elements of G. This follows from the fact any such fixed point is an attractor of the transformation itself (in the parabolic and loxodromic attractor case) or its inverse (in the case the fixed point is a repeller of a loxodromic element). Thus, if $z \in F \setminus E$, then the stabilizer subgroup G_z contains only the identity and elliptic transformations. This also holds for all $\theta_t(G_z)$, because as stated before, each θ_t is type preserving.

If $g, h \in G_z$ are nonidentity elements and do not have the same fixed point set, $ghg^{-1}h^{-1}$ is parabolic (see Section 9G in Chapter 2 of [**6**]). It follows that every element of G_z has the same two fixed points. The same is true for each $\theta_t(G_z)$. Since $\theta_t(g)$ depends holomorphically on t for each $g \in G$, for each $z \in F \setminus E$ there is a unique holomorphic function ψ_z on V such that $\psi_z(t_0) = z$ and $\psi_z(t)$ is fixed by $\theta_t(g)$ for all $g \in G_z$ and all $t \in V$.

We extend ϕ to $E \cup F$ by setting $\widetilde{\phi}(t, z) := \psi_z(t)$ if $t \in V$ and $z \in F \setminus E$. We claim this extended map is a G-equivariant holomorphic motion. For any $z \in F \cup E$, $\widetilde{\phi}(t_0, z) = z$ by construction. That $\widetilde{\phi}$ is holomorphic in the first coordinate also follows directly from construction.

Showing $\widetilde{\phi}$ is G-equivariant is only slightly more involved. Note $E \cup F$ is G-invariant; for E is G-invariant by hypothesis, and F is G-invariant by elementary algebra. If $z \in E$, $\widetilde{\phi}(t, g(z)) = (\theta_t(g))\widetilde{\phi}(t, z)$ for all g in G by hypothesis. If $z \in F \setminus E$, then the result follows from the definition of ψ_z and elementary facts about group actions.

The injectivity follows from the following

LEMMA 3.2. *If $\widetilde{\phi}(s, z) = \widetilde{\phi}(s, g(z))$ for some $g \in G, s \in V$ and some $z \in E \cup F$, then $g \in G_z$.*

The proof is given below.

We continue with the proof of Proposition 3.1. Suppose $\widetilde{\phi}(t,z)=\widetilde{\phi}(t,z')$, where $t\in V$ is fixed. We need to show $z=z'$. If both are in E, this is true by hypothesis. Assume, then, $z\in F\setminus E$. Then for all $g\in G$ we have

$$\widetilde{\phi}(t,g(z))=(\theta_t(g))(\widetilde{\phi}(t,z))=(\theta_t(g))(\widetilde{\phi}(t,z'))=\widetilde{\phi}(t,g(z')).$$

So if $g\in G_{z'}$, $\widetilde{\phi}(t,z)=\widetilde{\phi}(t,g(z))$. By Lemma 3.2, this implies that $g\in G_z$. Thus $G_{z'}\subset G_z$, and $G_z=G_{z'}$ follows because the argument is symmetric. Since $z\in F\setminus E$, G_z is a nontrivial group consisting only of elliptic elements all of which share the same fixed points. If $z\neq z'$, they must be these fixed points. So $\widetilde{\phi}(s,z')$ and $\widetilde{\phi}(s,z)$ are the two fixed points of $\theta_s(g)$ for any $s\in V$ and nontrivial $g\in G$ (this follows from the argument about disjoint graphs given in the proof of Lemma 3.2), contradicting our assumption $\widetilde{\phi}(t,z)=\widetilde{\phi}(t,z')$. So $z=z'$, and the proof is complete. □

Proof of Lemma 3.2. Assume there is some combination of g, z and s for which Lemma 3.2 is false. If $z\in E$ this cannot happen, so assume $z\in F\setminus E$ henceforth. For simplicity's sake let $w:=\widetilde{\phi}(s,z)$, and by our hypothesis and G-equivariance of $\widetilde{\phi}$ we have $\theta_s(g)(w)=w$. Choose a quasiconformal homeomorphism f_s of $\widehat{\mathbb{C}}$ inducing θ_s, and observe g fixes the point $z':=f_s^{-1}(w)$ because

$$g(z')=g\circ f_s^{-1}(w)=f_s^{-1}\circ f_s\circ g\circ f_s^{-1}(w)=f_s^{-1}\circ\theta_s(g)(w)=f_s^{-1}(w)=z'.$$

That is, $g\in G_{z'}$. If $z=z'$ there is nothing to prove, so we henceforth assume this is not the case. If $h\in G_z$, then by the G-equivariance we have

$$h(z')=h\circ f_s^{-1}(w)=f_s^{-1}\circ f_s\circ h\circ f_s^{-1}(w)=f_s^{-1}\theta_s(h)(w)=f_s^{-1}(w)=z'$$

implying that $G_z\subset G_{z'}$. Recall we assumed g was not in G_z, and choose a nontrivial $h\in G_z$. The commutator $h^*=hgh^{-1}g^{-1}$ is parabolic, so it can have only one fixed point, which will of course be z' since both g and h fix it. The transformation $\theta_s(h^*)$ is also parabolic, and it fixes $\widetilde{\phi}(s,z')$ by the G-equivariance and it fixes w because f_s induces θ_s. Therefore $\widetilde{\phi}(s,z')=w=\widetilde{\phi}(s,z)$. Since $h\in G_z$, and $G_z\subset G_{z'}$, G-equivariance implies $\theta_s(h)$ fixes both $\widetilde{\phi}(s,z')$ and $\widetilde{\phi}(s,z)$ for every $t\in V$. But $\theta_s(h)$ is always elliptic, and its fixed points are given by two holomorphic functions of s on V with disjoint graphs (as subsets of $V\times\widehat{\mathbb{C}}$).

It then follows from the definition of $\widetilde{\phi}$ that $\widetilde{\phi}(t,z)$ and $\widetilde{\phi}(t,z')$, as functions of t, either agree everywhere or agree nowhere. But we have already seen that when $t=s$, $\widetilde{\phi}(t,z)=\widetilde{\phi}(t,z')$. But this contradicts the fact $\widetilde{\phi}(t_0,z)\neq\widetilde{\phi}(t_0,z')$, since $z\neq z'$ by assumption. Hence we have a contradiction, and our lemma follows. □

4. Proof of Theorem B

We showed in the proof of Theorem A that the hypothesis has the implication that if A is *any* subset of $\widehat{\mathbb{C}}$, and $y\in\widehat{\mathbb{C}}\setminus A$, then there is an extension of ψ to $V\times(A\cup\{y\})$ which is also a holomorphic motion. Now, let ϕ and E be as in the hypothesis of our theorem, and let F be as in Proposition 3.1. Then ϕ has a G-equivariant extension to $V\times(E\cup F)$; denote this extension by ϕ as well for simplicity. Note that the definition of G-equivariance of a motion of a set clearly extends to the closure of that set. If $E\cup F$ is dense in $\widehat{\mathbb{C}}$, we are done, as ϕ extends to $V\times\widehat{\mathbb{C}}$ by Proposition 1.4. Otherwise let $\widetilde{E}$ be a G-invariant subset of $\widehat{\mathbb{C}}$ on which there is a G-equivariant holomorphic extension of ϕ, denoted by ϕ, again for simplicity, and further assume $(E \cup F) \subset \widetilde{E}$. Again, if $\widetilde{E}$ is dense in $\widehat{\mathbb{C}}$ we are done.

If not, take $y \in \widehat{\mathbb{C}} \setminus \widetilde{E}$, and extend ϕ to $\phi' : V \times (\widetilde{E} \cup \{y\})$. This can be done by the above comment. Now extend ϕ' to all of $V \times (\widetilde{E} \cup G(y))$ by the formula

$$\phi'(t, g(y)) := (\theta_t(g))(\phi'(t, y))$$

where $g \in G, t \in V$. Here $G(y)$ denotes the G-orbit of y, and this is well-defined because G_y is trivial (y is not in F). We claim this extended ϕ' is a G-equivariant holomorphic motion. Note that $\widetilde{E} \cup G(y)$ is G-invariant. Since $\theta_{t_0}(g) = g$, $\phi'(t_0, g(y)) := (\theta_{t_0}(g))(\varphi'(t_0, y)) = g(y)$, (i) of Definition 1.1 holds. Since for fixed g, $\theta_t(g)$ is holomorphic on $t \in V$, and $\phi'(t, y)$ is holomorphic on $t \in V$ by construction, for $g(y) \in G(y)$ we have $\phi'(t, g(y))$ is the product of two holomorphic functions, and so holomorphic itself. That ϕ' is G-equivariant is self-evident.

Before verifying injectivity, we make some general comments about fixed points of transformations in $\theta_t(G)$, where $t \in V$ is given. For any subset $D \subset E$ we define

$$\phi(t, D) := \{\zeta \in \widehat{\mathbb{C}} : \zeta = \phi(t, z) \text{ for some } z \in D\}.$$

For any nontrivial $g \in M\ddot{o}b$, let $Fix(g)$ be the set of fixed points of g. We claim that if $g \in G$, $\phi(t, Fix(g)) = Fix(\theta_t(g))$. Since θ_t is type-preserving, both $Fix(g)$ and $Fix(\theta_t(g))$ contain the same finite number of points. Now say $a \in Fix(g)$. Then $\phi(t, a) = \phi(t, g(a)) = (\theta_t(g))(\phi(t, a))$, implying $\phi(t, Fix(g)) \subset Fix(\theta_t(g))$, and equality follows.

Now, fix $t \in V$; we need to show $\phi'(t, z) = \phi'(t, z') \Rightarrow z = z'$. If both z and z' are in $\widetilde{E} \cup \{y\}$, this is true by construction. So assume $z \in \widetilde{E}$, and $z' \in G(y)$. There is a $g \in G$ such that $g(y) = z'$, and by G-invariance of $\widetilde{E}$ there is a $\zeta \in \widetilde{E}$ such that $g(\zeta) = z$. Then we have, by G-equivariance:

$$(\theta_t(g))(\phi'(t, \zeta)) = \phi'(t, z) = \phi'(t, z') = (\theta_t(g))(\phi'(t, y))$$

which implies that $\phi'(t, \zeta) = \phi'(t, y)$. Since the last statement is false, we have a contradiction.

Finally, assume both points are in $G(y)$, then there are distinct $g, h \in G$ such that $g(y) = z, h(y) = z'$, and $g \neq h$. Then $(\theta_t(g))(\phi'(t, y)) = (\theta_t(h))(\phi'(t, y))$. So $\theta_t(gh^{-1})$ fixes $\phi'(t, y)$. It follows from the above comments $y \in Fix(gh^{-1})$, implying $y \in F \subset \widetilde{E}$, a contradiction.

Step 2: Take Y a countable subset of $\widehat{\mathbb{C}} \setminus \widetilde{E}$ such that

(1) Any two distinct elements of Y are in distinct G-orbits.
(2) $\widetilde{E} \cup G(Y)$ is dense in $\widehat{\mathbb{C}}$, where $G(Y)$ is the G-orbit of the entire set Y.

By applying the logic in Step 1 repeatedly, we obtain a G-equivariant extension of ϕ to all of $V \times (\widetilde{E} \cup G(Y))$, and then apply Proposition 1.4. That completes the proof. □

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Second Edition, with additional chapters by C. J. Earle and I. Kra, M. Shishikura, J. H. Hubbard. American Mathematical Society. University Lecture Series, Volume 38 (2006). MR2241787 (2009d:30001)

[2] L. Bers and H. L. Royden, *Holomorphic families of injections,* Acta Math. **157** (1986), 259–286. MR857675 (88i:30034)

[3] C. J. Earle, *A Montel theorem for holomorphic functions on infinite dimensional spaces that omit the values 0 and 1*, Comput. Methods Funct. Theory **8** (2008), no. 1-2, 195-198. MR2419472 (2009f:46068)
[4] C. J. Earle, I. Kra, and S. L. Krushkal′, *Holomorphic motions and Teichmüller spaces,* Trans. Amer. Math. Soc. **343** (1994), 927-948. MR1214783 (94h:32035)
[5] Y. Jiang and S. Mitra, Some applications of universal holomorphic motions, Kodai Mathematical Journal, Vol. 30, No. 1 (2007), 85-96. MR2319079 (2008c:32019)
[6] J. Lehner, Discontinuous groups and automorphic functions; Math Surveys, VIII, American Mathematical Society, Providence, R.I., 1964. MR0164033 (29:1332)
[7] R. Mañe, P. Sad, and D. Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Sup. **16** (1983), 193-217. MR732343 (85j:58089)
[8] S. Mitra and H. Shiga, Extensions of holomorphic motions and holomorphic families of Möbius groups, Osaka Journal of Mathematics, **47** (2010), 1167-1187. MR2791561
[9] D. Sullivan and W. P. Thurston, *Extending holomorphic motions,* Acta Math. **157** (1986), 243-257. MR857674 (88i:30033)

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