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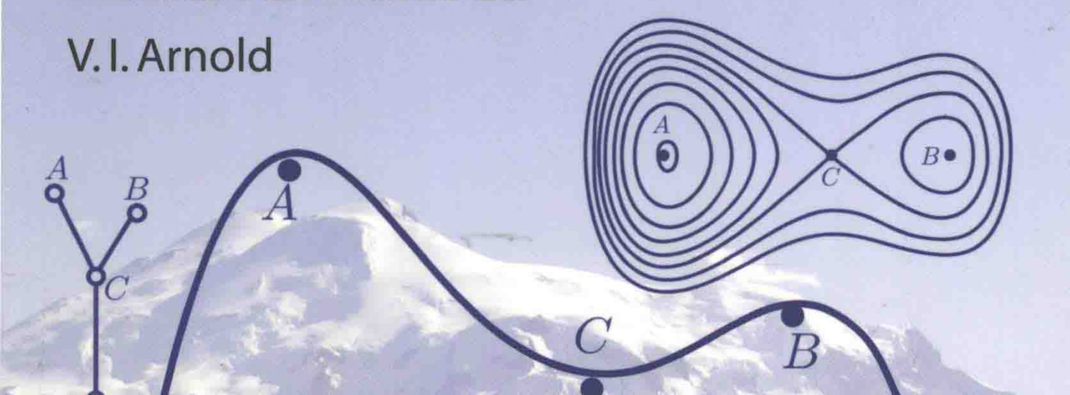
Experimental Mathematics

V. I. Arnold



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V. I. Arnold

Translated by Dmitry Fuchs and Mark Saul

MSRI
Mathematical Sciences Research Institute
Berkeley, California

AMS
American Mathematical Society
Providence, Rhode Island

Translated from Russian by Dmitry Fuchs and Mark Saul.

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Preface to the English Translation

Vladimir Arnold was one of the great mathematical minds of the late 20th century. His work was of great significance to the development of many areas of the field. On another level, Russian mathematicians have a strong tradition of writing for, and even directly teaching, younger students interested in mathematics. This work is an example of Arnold's contributions to the genre.

In 2005, Arnold gave lectures at the Dubna summer camp. This camp is an extraordinary gathering of the Russian mathematical community, in which distinguished mathematicians work to support advanced high school and undergraduate students entering the field. The present book is based on notes from these lectures. As the reader will see, Arnold was very connected to the new generation of mathematicians. One can sense the urgency he felt at delivering his thoughts into hands that might take them farther. The reader expecting a formal mathematical exposition will sometimes not find it here.

One might mistake this style of the work as not just urgent, but sloppy. No. The style is well thought out. Arnold's approach to mathematics—and he makes this quite clear in several passages—was fluid and intuitive. He saw mathematics not as a flat plain to be surveyed, but as a rugged terrain to explore. The most exciting aspect of mathematics, for Arnold, seems to have been a dynamic search for pattern through examination of many special cases. That is, he held a severely Platonic view of the subject, as one that proceeds as if it were an experimental science—hence the title. After this exploratory phase, on can tuck in the ragged edges. Arnold does this in many—but not all—cases, giving us theorems and proofs in the classic manner.

But it is in the chase, in the experimental "phase" of the process of doing mathematics, that Arnold here seems to take the most joy, and offers this joy to a new generation. Mathematical mainstream culture, in which one burns one's scrap work, discourages this. Few mathematicians—indeed few scientists in any field—open their minds so completely as he has to their students.

Arnold's style is unforgiving. The reader—even the professional mathematician—will find paragraphs that require hours of thought to unscramble. In some cases, Arnold collapses an argument into a few sentences that might take up several pages in another style of exposition. In other cases, he gives an intuitive argument in place of a rigorous one, leaving the reader to construct the latter. He probably felt that the real work was done on the intuitive level, and that his teaching would be the more effective if he left the tidying up to the student. The reader must have patience with the ellipses of thought and the leaps of reason. They are all part of Arnold's intent.

These lecture notes were gathered in haste from the field, and we have corrected numerous misprints and small errors in notation. We have given several extensions—in Arnold's own style—to the work, in "editors' notes". At the same time, we have striven to deliver intact the style of the work. Arnold's mind leaps from peak to peak, connecting disparate areas of mathematics, all (or most) accessible to the student with an advanced high school education. And yet there is a unity to each lecture, a flow from very simple questions to deep intellectual inquiry, and sometimes right to the edge of our knowledge of mathematics.

We hope that we have preserved this coherence, but also the excitement of the work, the sharp, jagged edges and breathtaking jumps that characterize the author's thinking.

It is our pleasure to acknowledge the contributions of several colleagues to this work. In particular, Sergei Gelfand, at the American Mathematical Society, kept us on track at several key junctures. James Fennell sedulously proofread the manuscript and corrected the TEX files. We would also like to thank the students of the Gradus ad Parnassum math circles at the Courant Institute, who gave us feedback about several sections. Much of this work was supported by a generous grant from the Alfred P. Sloan Foundation.

MARK SAUL
February 2015

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Introduction

> "Not having achieved what they desired, they pretended to desire what they had achieved."
>
> *M. Montel*

In this series of lectures I will talk about several new directions in mathematical research. All of these are based on the idea of numerical experimentation. After looking at examples such as $5 \cdot 5 = 25$ and $6 \cdot 6 = 36$, we advance an hypothesis, such as $7 \cdot 7 = 47$. Further experimentation either supports or disproves it.

For example, Fermat's hypothesis (that the equation $x^n + y^n = z^n$ cannot be solved in natural numbers with $n > 2$) was advanced as a result of his attempts at a solution. This hypothesis led to the creation of a whole field of knowledge, but it was proved only after a few hundred years had passed.

The majority of hypotheses that we make are so far not proven (nor refuted). I decided to give these lectures exactly because of my hope that the listeners will help in the investigation of these problems, if only by conducting numerical experiments (which I have also conducted, without a computer, in the bounded region of the first million numbers).

Lecture 1

The Statistics of Topology and Algebra

> "I never heard of such a mathematician: he is actually a physicist."
>
> *Landau, on Poincaré*

> "It is not Shakespeare that most matters, but commentaries on his work."
>
> *A. P. Chekhov, as described by B. L. Pasternak*

Poincaré, the greatest mathematician of the recent era, divided all problems into two classes: binary problems and interesting problems. Binary problems are problems which admit of an answer "yes" or "no" (for example, Fermat's question).

Interesting problems are those for which an answer of "yes" or "no" is insufficient. They require investigation of questions that lead one further. For example, Poincaré was interested in how to change the conditions of a problem (for instance, the boundary conditions of a differential equation), while retaining the existence and uniqueness of its solution, or how the number of solutions varies when we make some other change. Thus he started the theory of bifurcations.

Three years before Hilbert gave his list of problems, Poincaré formulated the basic, in his view, mathematical questions that the nineteenth century would leave for the twentieth. This was the formulation of the mathematical basis for quantum and relativistic physics.

Today, many people think that relativistic physics at the time, in 1897, did not yet exist, since Einstein published his theory of relativity only in 1905. But Poincaré formulated the principle of relativity earlier, in his article of 1895, "On the Measurement of Time", which Einstein actually used (and which, by the way, he didn't acknowledge in writing until 1945). In just the same way, Schrödinger, in laying the foundation for quantum mechanics, achieved his success only because he used the mathematical works of his predecessor Hermann Weyl, whom no one mentioned later on, although Schrödinger actually references these works (in his first book).

1. Hilbert's Sixteenth Problem

Although I basically agree with Poincaré, today I will talk about a binary problem (or almost binary: this is why I am going to talk about of it) posed by Hilbert, the 16th in his list.

This problem is actually much older than Hilbert's list. In general, it is one of the fundamental problems of all of mathematical science (and of many of its applications).

Here is a very simple example: for an algebraic polynomial f in two variables x and y, we look at the curve along which it equals zero:

$$\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$$

The problem consists in determining *the possible topological structures of this curve, if f is a polynomial of a given degree n.*

For example, if $n = 2$, then by the ancient theory of conic sections, the curve can be an ellipse, a hyperbola, a parabola, a pair of lines (which might possibly coincide), or the entire plane (if the polynomial is identically 0).

Augmenting the plane with points at infinity turns it into the projective plane, for which the problem becomes easier. (An ellipse, a hyperbola, and a parabola have the same structure in the projective plane. The only difference is in the position of this "circle" with respect to the line at infinity. See Figure 1.)

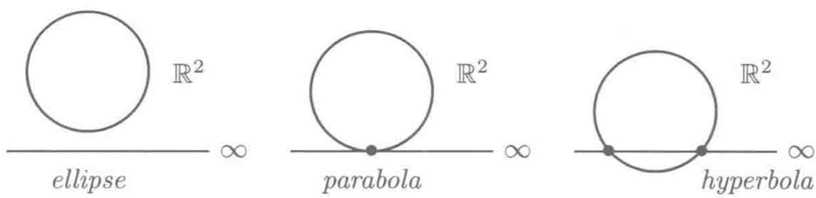

Figure 1

For $n > 2$ the question is more difficult, but Descartes and Newton had already analyzed the cases $n = 3$ and $n = 4$. Hilbert asserted the he had looked into it for curves of degree $n = 6$, but his result (the proof of which he never published) was erroneous.

According to a theorem of Harnack, a curve of degree n consists of no more than $g + 1 = \dfrac{(n-1)(n-2)}{2} + 1$ connected components (where g is the genus of the associated Riemann surface formed by the complex solutions of the equation of the curve in the complex projective plane $\mathbb{C}P^2$). According to a theorem in topology, every closed connected orientable surface is a surface of genus g, where g is the number of handles we must affix to a sphere in order to obtain this surface (see Figure 2).

For $n = 6$ we find that the genus of the Riemann surface g is 10, so that a real curve of degree 6 has no more than 11 components (which are called "ovals", and resemble circles, or at least are diffeomorphic to the circle S^1).

Hilbert asserted that if the number of ovals is maximal (that is, if there are 11 ovals), then these 11 ovals can be placed on the (projective) plane $\mathbb{R}P^2$ *in only two ways.*

Each oval bounds a "disk", diffeomorphic to the interior of a circle. (The complement of this disk in $\mathbb{R}P^2$ forms a Möbius band: this is how Möbius discovered his surface.)

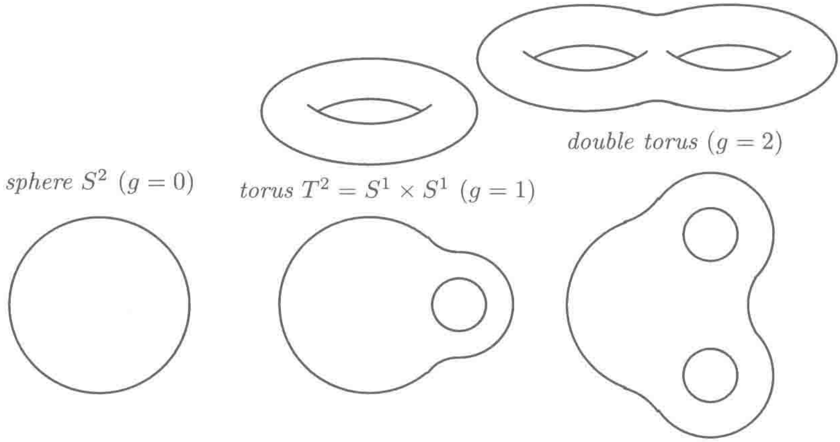

Figure 2. Surfaces of genus 0, genus 1, and genus 2.

And so, Hilbert asserted that only one of these disks can contain any other ovals inside it, and the number of interior ovals can only take on two values: 1 and 9 (Figure 3).

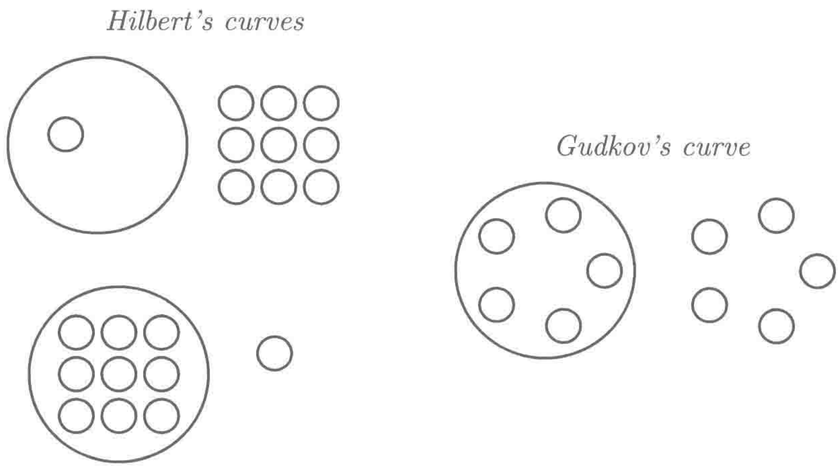

Figure 3. Algebraic curve of degree 6 with 11 ovals.

Hilbert's error consisted in the fact that the number of interior ovals could also be equal to 5. (This was discovered by Dmitri Andreevich Gudkov, a mathematician from Nizhny Novgorod, around 1970.) (See Editor's note 1, page 55.)

For curves of degree 8 Hilbert's question remains unanswered to this day: the 22 ovals of a curve of degree 8 can be placed on the plane in billions of different ways. But now certain bounds have been found which reduce the number of topologically distinct curves. There are now fewer than 90 cases. However, the number of examples actually constructed, while greater than 70, is not as large as the number of theoretical possibilities.

It is interesting that although the question seems to concern computational mathematics, our computers, so far, have contributed almost nothing to its solution.

If the coefficients of a polynomial are known, then it is possible for a computer to draw the positions of the ovals corresponding to the curve. But a count of *all* possibilities (for any values of the coefficients) is a much more difficult problem.

The problem also has an algorithmic solution, in the sense of mathematical logic. In principle, we can even find the number of connected regions into which the space of polynomials of degree n is divided by a bifurcation diagram, near which the type of the curve changes. But the number of computations needed for this is so large that no progress in computer technology will allow us the hope of a computer solution for the problem of polynomials of degree 8 in the foreseeable future.

Drifting a bit from the theme of today's lecture, I shall talk about one very recent success of computer technology that I know of, with regard to a closely related problem.

Let us think of the graph of a real polynomial of degree n in two variables as a surface, $z = f(x, y)$ in three dimensional space $\mathbb{R}^3$. Near some of its points the surface is locally convex. We call such points *elliptical* points. Around other points, the surface is locally saddle-shaped. We call these points *hyperbolic* points (see Figure 4).

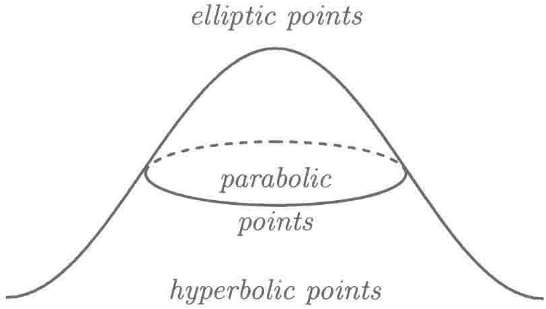

Figure 4. The parabolic curve on a smooth surface.

The elliptical and hyperbolic points are divided by a curve consisting of *parabolic points*. In terms of the partial derivatives of the function f, the curve of parabolic points is given by the equation

$$\det \begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix} = 0,$$

that is, by the condition $f_{xx} f_{yy} = (f_{xy})^2$, or that the Hessian of the function f is zero.

Let f be a polynomial of degree n. We may ask: *how many closed curves (ovals) can its parabolic curve be made up of?*

For a polynomial f of degree 4, the Hessian is also of degree 4, so by Harnack's theorem the number of ovals cannot exceed $g + 1 = 4$.

It is not hard to construct a polynomial of degree 4, with a parabolic curve consisting of three ovals. I leave this problem as an exercise for the reader.

But the problem of whether the parabolic curve of a polynomial of degree 4 can consist of four ovals turns out to be very difficult.

It was solved in Mexico in 2005 by Adriana Ortiz-Rodriguez, who defended her dissertation in Paris as my student. In her dissertation, she proved that the number of ovals in the parabolic curve of a polynomial of degree n is bounded from above by an^2 and below by bn^2, where $a > b$.