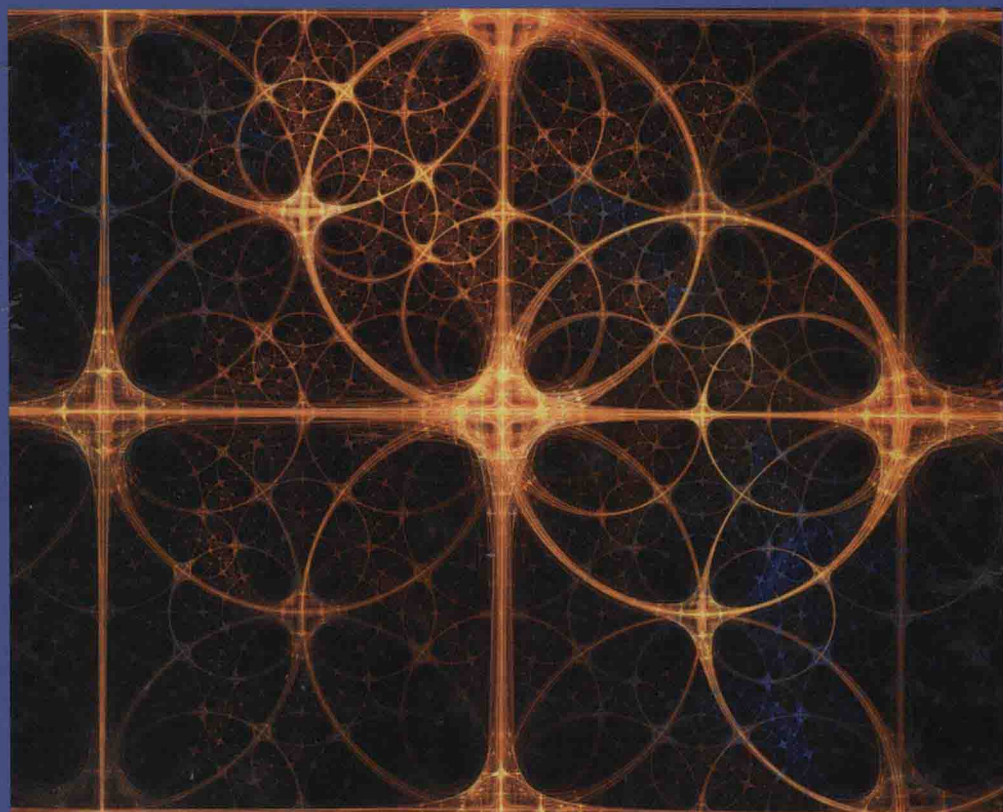


TEXTBOOKS in MATHEMATICS

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

Revised Edition



Lawrence C. Evans
Ronald F. Gariepy



CRC Press
Taylor & Francis Group

A CHAPMAN & HALL BOOK

TEXTBOOKS in MATHEMATICS

0174.12

E92M

REV.

2015

(复印件)

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

Revised Edition

Lawrence C. Evans

University of California
Berkeley, USA

Ronald F. Gariepy

University of Kentucky
Lexington, USA



CRC Press

Taylor & Francis Group

Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group an Informa business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

© 2015 by Taylor & Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works

Printed on acid-free paper
Version Date: 20150317

International Standard Book Number-13: 978-1-4822-4238-6 (Hardback)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (<http://www.copyright.com/>) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Visit the Taylor & Francis Web site at
<http://www.taylorandfrancis.com>

and the CRC Press Web site at
<http://www.crcpress.com>

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

Revised Edition

PUBLISHED TITLES CONTINUED

EXPLORING LINEAR ALGEBRA: LABS AND PROJECTS WITH MATHEMATICA®

Crista Arangala

AN INTRODUCTION TO NUMBER THEORY WITH CRYPTOGRAPHY

James Kraft and Larry Washington

AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS WITH MATLAB®, SECOND EDITION

Mathew Coleman

INTRODUCTION TO THE CALCULUS OF VARIATIONS AND CONTROL WITH MODERN APPLICATIONS

John T. Burns

INTRODUCTION TO MATHEMATICAL LOGIC, SIXTH EDITION

Elliott Mendelson

INTRODUCTION TO MATHEMATICAL PROOFS: A TRANSITION TO ADVANCED MATHEMATICS, SECOND EDITION

Charles E. Roberts, Jr.

LINEAR ALGEBRA, GEOMETRY AND TRANSFORMATION

Bruce Solomon

THE MATHEMATICS OF GAMES: AN INTRODUCTION TO PROBABILITY

David G. Taylor

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS, REVISED EDITION

Lawrence C. Evans and Ronald F. Gariepy

QUADRATIC IRRATIONALS: AN INTRODUCTION TO CLASSICAL NUMBER THEORY

Franz Holter-Koch

REAL ANALYSIS AND FOUNDATIONS, THIRD EDITION

Steven G. Krantz

RISK ANALYSIS IN ENGINEERING AND ECONOMICS, SECOND EDITION

Bilal M. Ayyub

RISK MANAGEMENT AND SIMULATION

Aparna Gupta

TRANSFORMATIONAL PLANE GEOMETRY

Ronald N. Umble and Zhigang Han

Preface to the Revised Edition

We published the original edition of this book in 1992 and have been extremely gratified with its popularity for now over 20 years. The publisher recently asked us to write an update, and we have agreed to do so in return for a promise that the future price be kept reasonable.

For this revised edition the entire book has been retyped into LaTeX and we have accordingly been able to set up better cross-references with page numbers. There have been countless improvements in notation, format and clarity of exposition, and the bibliography has been updated. We have also added several new sections, describing the π - λ Theorem, weak compactness criteria in L^1 and Young measure methods for weak convergence.

We will post any future corrections or comments at LCE's homepage, accessible through the math.berkeley.edu website. We remain very grateful to the many readers who have written us over the years, suggesting improvements and error fixes.

LCE has been supported during the writing of the revised edition by the National Science Foundation (under the grant DMS-1301661), by the Miller Institute for Basic Research in Science and by the Class of 1961 Collegium Chair at UC Berkeley.

Best wishes to our readers, past and future.

LCE/RFG
November, 2014
Berkeley/Lexington

Preface

These notes gather together what we regard as the essentials of real analysis on \mathbb{R}^n .

There are of course many good texts describing, on the one hand, Lebesgue measure for the real line and, on the other, general measures for abstract spaces. But we believe there is still a need for a source book documenting the rich structure of measure theory on \mathbb{R}^n , with particular emphasis on integration and differentiation. And so we packed into these notes all sorts of interesting topics that working mathematical analysts need to know, but are mostly not taught. These include Hausdorff measures and capacities (for classifying “negligible” sets for various fine properties of functions), Rademacher’s Theorem (asserting the differentiability of Lipschitz continuous functions almost everywhere), Aleksandrov’s Theorem (asserting the twice differentiability of convex functions almost everywhere), the area and coarea formulas (yielding change-of-variables rules for Lipschitz continuous maps between \mathbb{R}^n and \mathbb{R}^m), and the Lebesgue–Besicovitch Differentiation Theorem (amounting to the fundamental theorem of calculus for real analysis).

This book is definitely not for beginners. We explicitly assume our readers are at least fairly conversant with both Lebesgue measure and abstract measure theory. The expository style reflects this expectation. We do not offer lengthy heuristics or motivation, but as compensation have tried to present *all* the technicalities of the proofs: “God is in the details.”

Chapter 1 comprises a quick review of mostly standard real analysis, Chapter 2 introduces Hausdorff measures, and Chapter 3 discusses the area and coarea formulas. In Chapters 4 through 6 we analyze the fine properties of functions possessing weak derivatives of various sorts. Sobolev functions, which is to say functions having weak first partial derivatives in an L^p space, are the subject of Chapter 4; functions of bounded variation, that is, functions having measures as weak first partial derivatives, the subject of Chapter 5. Finally, Chapter 6 discusses

the approximation of Lipschitz continuous, Sobolev and BV functions by C^1 functions, and several related subjects.

We have listed in the references the primary sources we have relied upon for these notes. In addition many colleagues, in particular S. Antman, J.-A. Cohen, M. Crandall, A. Damlamian, H. Ishii, N.V. Krylov, N. Owen, P. Souganidis, S. Spector, and W. Strauss, have suggested improvements and detected errors. We have also made use of S. Katzenburger's class notes. Early drafts of the manuscript were typed by E. Hampton, M. Hourihan, B. Kaufman, and J. Slack.

LCE was partially supported by NSF Grants DMS-83-01265, 86-01532, and 89-03328, and by the Institute for Physical Science and Technology at the University of Maryland. RFG was partially supported by NSF Grant DMS-87-04111 and by NSF Grant RII-86-10671 and the Commonwealth of Kentucky through the Kentucky EPSCoR program.

Warnings

Our terminology is occasionally at variance with standard usage. The principal changes are these:

- What we call a *measure* is usually called an *outer measure*.
- For us a function is *integrable* if it has an integral (which may equal $\pm\infty$).
- We call a function f *summable* if $|f|$ has a finite integral.
- We do *not* identify two L^p , BV or Sobolev functions that agree almost everywhere.

Contents

Preface to the Revised Edition	xi
Preface	xiii
1 General Measure Theory	1
1.1 Measures and measurable functions	1
1.1.1 Measures	1
1.1.2 Systems of sets	5
1.1.3 Approximation by open and compact sets	9
1.1.4 Measurable functions	16
1.2 Lusin's and Egoroff's Theorems	19
1.3 Integrals and limit theorems	24
1.4 Product measures, Fubini's Theorem, Lebesgue measure	29
1.5 Covering theorems	35
1.5.1 Vitali's Covering Theorem	35
1.5.2 Besicovitch's Covering Theorem	39
1.6 Differentiation of Radon measures	47
1.6.1 Derivatives	47
1.6.2 Integration of derivatives; Lebesgue decomposition	50
1.7 Lebesgue points, approximate continuity	53
1.7.1 Differentiation Theorem	53
1.7.2 Approximate limits, approximate continuity	56
1.8 Riesz Representation Theorem	59
1.9 Weak convergence	65
1.9.1 Weak convergence of measures	65
1.9.2 Weak convergence of functions	68
1.9.3 Weak convergence in L^1	70
1.9.4 Measures of oscillation	75
1.10 References and notes	78

2	Hausdorff Measures	81
2.1	Definitions and elementary properties	81
2.2	Isodiametric inequality, $\mathcal{H}^n = \mathcal{L}^n$	87
2.3	Densities	92
2.4	Functions and Hausdorff measure	96
2.4.1	Hausdorff measure and Lipschitz mappings . . .	96
2.4.2	Graphs of Lipschitz functions	97
2.4.3	Integrals over balls	98
2.5	References and notes	100
3	Area and Coarea Formulas	101
3.1	Lipschitz functions, Rademacher's Theorem	101
3.1.1	Lipschitz continuous functions	101
3.1.2	Rademacher's Theorem	103
3.2	Linear maps and Jacobians	108
3.2.1	Linear mappings	108
3.2.2	Jacobians	114
3.3	The area formula	114
3.3.1	Preliminaries	114
3.3.2	Proof of the area formula	119
3.3.3	Change of variables formula	122
3.3.4	Applications	123
3.4	The coarea formula	126
3.4.1	Preliminaries	126
3.4.2	Proof of the coarea formula	134
3.4.3	Change of variables formula	139
3.4.4	Applications	140
3.5	References and notes	142
4	Sobolev Functions	143
4.1	Definitions and elementary properties	143
4.2	Approximation	145
4.2.1	Approximation by smooth functions	145
4.2.2	Product and chain rules	153
4.2.3	$W^{1,\infty}$ and Lipschitz continuous functions	155
4.3	Traces	156
4.4	Extensions	158

4.5	Sobolev inequalities	162
4.5.1	Gagliardo–Nirenberg–Sobolev inequality	162
4.5.2	Poincaré’s inequality on balls	164
4.5.3	Morrey’s inequality	167
4.6	Compactness	168
4.7	Capacity	170
4.7.1	Definitions and elementary properties	171
4.7.2	Capacity and Hausdorff dimension	179
4.8	Quasicontinuity, precise representatives of Sobolev func- tions	183
4.9	Differentiability on lines	187
4.9.1	Sobolev functions of one variable	188
4.9.2	Differentiability on a.e. line	189
4.10	References and notes	190

5 Functions of Bounded Variation, Sets of Finite Perimeter 193

5.1	Definitions, Structure Theorem	193
5.2	Approximation and compactness	199
5.2.1	Lower semicontinuity	199
5.2.2	Approximation by smooth functions	199
5.2.3	Compactness	203
5.3	Traces	204
5.4	Extensions	210
5.5	Coarea formula for BV functions	212
5.6	Isoperimetric inequalities	215
5.6.1	Sobolev’s and Poincaré’s inequalities for BV	216
5.6.2	Isoperimetric inequalities	217
5.6.3	\mathcal{H}^{n-1} and Cap_1	220
5.7	The reduced boundary	221
5.7.1	Estimates	221
5.7.2	Blow-up	225
5.7.3	Structure Theorem for sets of finite perimeter	231
5.8	Gauss–Green Theorem	235
5.9	Pointwise properties of BV functions	236
5.10	Essential variation on lines	244
5.10.1	BV functions of one variable	245
5.10.2	Essential variation on almost all lines	247
5.11	A criterion for finite perimeter	249

5.12	References and notes	255
6	Differentiability, Approximation by C^1 Functions	257
6.1	L^p differentiability, approximate differentiability	257
6.1.1	L^{1*} differentiability for BV	257
6.1.2	L^{p*} differentiability a.e. for $W^{1,p}$	260
6.1.3	Approximate differentiability	262
6.2	Differentiability a.e. for $W^{1,p}$ ($p > n$)	265
6.3	Convex functions	266
6.4	Second derivatives a.e. for convex functions	273
6.5	Whitney's Extension Theorem	276
6.6	Approximation by C^1 functions	282
6.6.1	Approximation of Lipschitz continuous functions	283
6.6.2	Approximation of BV functions	283
6.6.3	Approximation of Sobolev functions	286
6.7	References and notes	288
	Bibliography	289
	Notation	293
	Index	297

Chapter 1

General Measure Theory

This chapter is mostly a review of standard measure theory, with particular attention paid to Radon measures on \mathbb{R}^n .

Sections 1.1 through 1.4 are a rapid recounting of abstract measure theory. In Section 1.5 we establish Vitali's and Besicovitch's Covering Theorems, the latter being the key for the Lebesgue–Besicovitch Differentiation Theorem for Radon measures in Sections 1.6 and 1.7. Section 1.8 provides a vector-valued version of Riesz's Representation Theorem. In Section 1.9 we study weak compactness for sequences of measures and functions.

The reader should as necessary consult the Appendix for a summary of our notation.

1.1 Measures and measurable functions

1.1.1 Measures

Although we intend later to work almost exclusively in \mathbb{R}^n , it is most convenient to start abstractly.

Let X denote a nonempty set, and 2^X the collection of all subsets of X .

DEFINITION 1.1. *A mapping $\mu : 2^X \rightarrow [0, \infty]$ is called a **measure** on X provided*

(i) $\mu(\emptyset) = 0$, and

(ii) if

$$A \subseteq \bigcup_{k=1}^{\infty} A_k,$$

then

$$\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Condition (ii) is called *subadditivity*.

Warning: Most texts call such a mapping μ an *outer measure*, reserving the name *measure* for μ restricted to the collection of μ -measurable subsets of X (see below). We will see, however, that there are definite advantages to being able to “measure” even nonmeasurable sets.

DEFINITION 1.2. Let μ be a measure on X and $C \subseteq X$. Then μ restricted to C , written

$$\mu \llcorner C,$$

is the measure defined by

$$(\mu \llcorner C)(A) := \mu(A \cap C) \quad \text{for all } A \subseteq X.$$

DEFINITION 1.3. A set $A \subseteq X$ is μ -measurable if for each set $B \subseteq X$ we have

$$\mu(B) = \mu(B \cap A) + \mu(B - A).$$

THEOREM 1.1 (Elementary properties of measures). Let μ be a measure on X .

(i) If $A \subseteq B \subseteq X$, then

$$\mu(A) \leq \mu(B).$$

(ii) A set A is μ -measurable if and only if $X - A$ is μ -measurable.

(iii) The sets \emptyset and X are μ -measurable. More generally, if $\mu(A) = 0$, then A is μ -measurable.

(iv) If C is any subset of X , then each μ -measurable set is also $\mu \llcorner C$ -measurable.

Proof. 1. Assertion (i) follows at once from the definition. To show (ii), assume A is μ -measurable and $B \subseteq X$. Then

$$\mu(B) = \mu(B \cap A) + \mu(B - A) = \mu(B - (X - A)) + \mu(B \cap (X - A));$$

and so $X - A$ is μ -measurable.

2. Suppose now $\mu(A) = 0$, $B \subseteq X$. Then $\mu(B \cap A) = 0$, and consequently

$$\mu(B) \geq \mu(B - A) = \mu(B \cap A) + \mu(B - A).$$

The opposite inequality is clear from subadditivity.

3. Assume A is μ -measurable, $B \subseteq X$. Then

$$\begin{aligned} \mu \llcorner C(B) &= \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) - A) \\ &= \mu((B \cap A) \cap C) + \mu((B - A) \cap C) \\ &= \mu \llcorner C(B \cap A) + \mu \llcorner C(B - A). \end{aligned}$$

Hence A is $\mu \llcorner C$ -measurable. □

THEOREM 1.2 (Sequences of measurable sets). *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ -measurable sets.*

(i) *The sets*

$$\bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k$$

are μ -measurable.

(ii) *If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint, then*

$$\mu \left(\bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu(A_k).$$

(iii) *If $A_1 \subseteq \dots \subseteq A_k \subseteq A_{k+1} \dots$, then*

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu \left(\bigcup_{k=1}^{\infty} A_k \right).$$

(iv) *If $A_1 \supset \dots \supset A_k \supset A_{k+1} \dots$ and $\mu(A_1) < \infty$, then*

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu \left(\bigcap_{k=1}^{\infty} A_k \right).$$