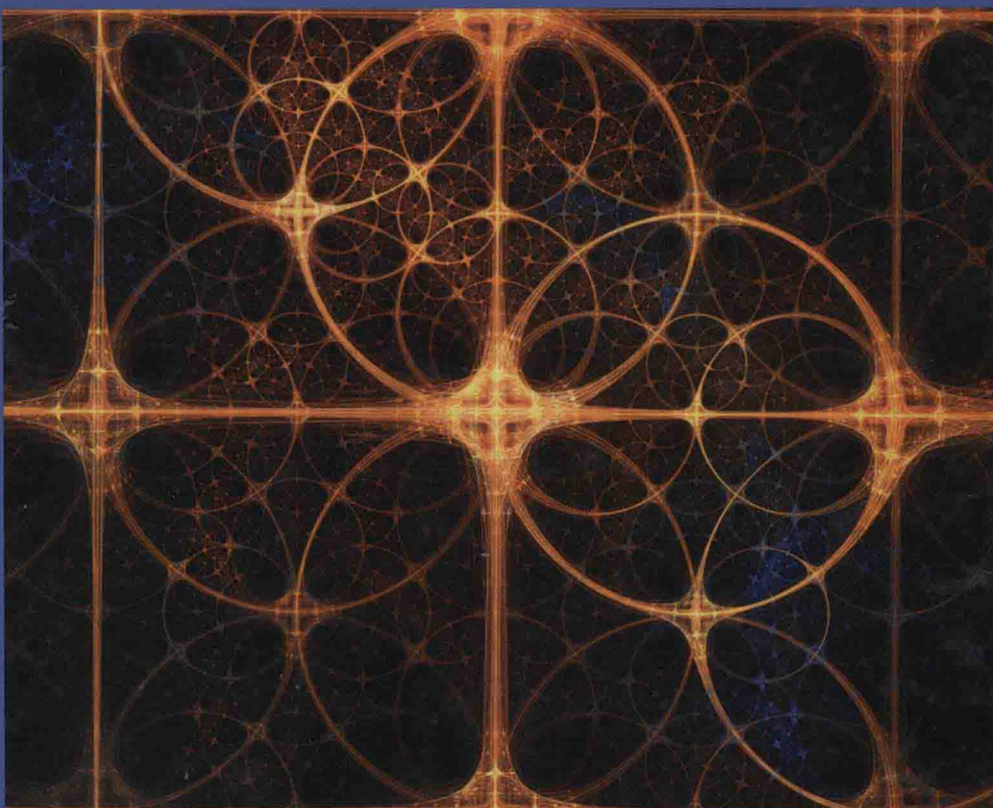
TEXTBOOKS in MATHEMATICS

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS

Revised Edition



Lawrence C. Evans

Ronald F. Gariepy

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A CHAPMAN & HALL BOOK

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Preface to the Revised Edition

We published the original edition of this book in 1992 and have been extremely gratified with its popularity for now over 20 years. The publisher recently asked us to write an update, and we have agreed to do so in return for a promise that the future price be kept reasonable.

For this revised edition the entire book has been retyped into LaTeX and we have accordingly been able to set up better cross-references with page numbers. There have been countless improvements in notation, format and clarity of exposition, and the bibliography has been updated. We have also added several new sections, describing the π-λ Theorem, weak compactness criteria in L^1 and Young measure methods for weak convergence.

We will post any future corrections or comments at LCE's homepage, accessible through the math.berkeley.edu website. We remain very grateful to the many readers who have written us over the years, suggesting improvements and error fixes.

LCE has been supported during the writing of the revised edition by the National Science Foundation (under the grant DMS-1301661), by the Miller Institute for Basic Research in Science and by the Class of 1961 Collegium Chair at UC Berkeley.

Best wishes to our readers, past and future.

LCE/RFG
November, 2014
Berkeley/Lexington

Preface

These notes gather together what we regard as the essentials of real analysis on $\mathbb{R}^n$.

There are of course many good texts describing, on the one hand, Lebesgue measure for the real line and, on the other, general measures for abstract spaces. But we believe there is still a need for a source book documenting the rich structure of measure theory on $\mathbb{R}^n$, with particular emphasis on integration and differentiation. And so we packed into these notes all sorts of interesting topics that working mathematical analysts need to know, but are mostly not taught. These include Hausdorff measures and capacities (for classifying "negligible" sets for various fine properties of functions), Rademacher's Theorem (asserting the differentiability of Lipschitz continuous functions almost everywhere), Aleksandrov's Theorem (asserting the twice differentiability of convex functions almost everywhere), the area and coarea formulas (yielding change-of-variables rules for Lipschitz continuous maps between $\mathbb{R}^n$ and $\mathbb{R}^m$), and the Lebesgue–Besicovitch Differentiation Theorem (amounting to the fundamental theorem of calculus for real analysis).

This book is definitely not for beginners. We explicitly assume our readers are at least fairly conversant with both Lebesgue measure and abstract measure theory. The expository style reflects this expectation. We do not offer lengthy heuristics or motivation, but as compensation have tried to present *all* the technicalities of the proofs: "God is in the details."

Chapter 1 comprises a quick review of mostly standard real analysis, Chapter 2 introduces Hausdorff measures, and Chapter 3 discusses the area and coarea formulas. In Chapters 4 through 6 we analyze the fine properties of functions possessing weak derivatives of various sorts. Sobolev functions, which is to say functions having weak first partial derivatives in an L^p space, are the subject of Chapter 4; functions of bounded variation, that is, functions having measures as weak first partial derivatives, the subject of Chapter 5. Finally, Chapter 6 discusses the approximation of Lipschitz continuous, Sobolev and BV functions by C^1 functions, and several related subjects.

We have listed in the references the primary sources we have relied upon for these notes. In addition many colleagues, in particular S. Antman, J.-A. Cohen, M. Crandall, A. Damlamian, H. Ishii, N.V. Krylov, N. Owen, P. Souganidis, S. Spector, and W. Strauss, have suggested improvements and detected errors. We have also made use of S. Katzenburger's class notes. Early drafts of the manuscript were typed by E. Hampton, M. Hourihan, B. Kaufman, and J. Slack.

LCE was partially supported by NSF Grants DMS-83-01265, 86-01532, and 89-03328, and by the Institute for Physical Science and Technology at the University of Maryland. RFG was partially supported by NSF Grant DMS-87-04111 and by NSF Grant RII-86-10671 and the Commonwealth of Kentucky through the Kentucky EPSCoR program.

Warnings

Our terminology is occasionally at variance with standard usage. The principal changes are these:

- What we call a *measure* is usually called an *outer measure*.
- For us a function is *integrable* if it has an integral (which may equal $\pm\infty$).
- We call a function f *summable* if $|f|$ has a finite integral.
- We do *not* identify two L^p, BV or Sobolev functions that agree almost everywhere.

Contents

Chapter 1

General Measure Theory

This chapter is mostly a review of standard measure theory, with particular attention paid to Radon measures on $\mathbb{R}^n$.

Sections 1.1 through 1.4 are a rapid recounting of abstract measure theory. In Section 1.5 we establish Vitali's and Besicovitch's Covering Theorems, the latter being the key for the Lebesgue–Besicovitch Differentiation Theorem for Radon measures in Sections 1.6 and 1.7. Section 1.8 provides a vector-valued version of Riesz's Representation Theorem. In Section 1.9 we study weak compactness for sequences of measures and functions.

The reader should as necessary consult the Appendix for a summary of our notation.

1.1 Measures and measurable functions

1.1.1 Measures

Although we intend later to work almost exclusively in $\mathbb{R}^n$, it is most convenient to start abstractly.

Let X denote a nonempty set, and 2^X the collection of all subsets of X.

DEFINITION 1.1. *A mapping $\mu : 2^X \to [0, \infty]$ is called a* **measure** *on X provided*

(i) $\mu(\emptyset) = 0$, *and*

(ii) *if*

$$A \subseteq \bigcup_{k=1}^{\infty} A_k,$$

then

$$\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Condition (ii) is called *subadditivity*.

Warning: Most texts call such a mapping μ an *outer measure*, reserving the name *measure* for μ restricted to the collection of μ-measurable subsets of X (see below). We will see, however, that there are definite advantages to being able to "measure" even nonmeasurable sets.

DEFINITION 1.2. *Let μ be a measure on X and $C \subseteq X$. Then μ* **restricted to** C*, written*

$$\mu \llcorner C,$$

is the measure defined by

$$(\mu \llcorner C)(A) := \mu(A \cap C) \quad \textit{for all } A \subseteq X.$$

DEFINITION 1.3. *A set $A \subseteq X$ is* **μ-measurable** *if for each set $B \subseteq X$ we have*

$$\mu(B) = \mu(B \cap A) + \mu(B - A).$$

THEOREM 1.1 (Elementary properties of measures). *Let μ be a measure on X.*

(i) *If $A \subseteq B \subseteq X$, then*

$$\mu(A) \leq \mu(B).$$

(ii) *A set A is μ-measurable if and only if $X - A$ is μ-measurable.*

(iii) *The sets $\emptyset$ and X are μ-measurable. More generally, if $\mu(A) = 0$, then A is μ-measurable.*

(iv) *If C is any subset of X, then each μ-measurable set is also $\mu \llcorner C$-measurable.*

Proof. 1. Assertion (i) follows at once from the definition. To show (ii), assume A is μ-measurable and $B \subseteq X$. Then

$$\mu(B) = \mu(B \cap A) + \mu(B - A) = \mu(B - (X - A)) + \mu(B \cap (X - A));$$

and so $X - A$ is μ-measurable.

2. Suppose now $\mu(A) = 0$, $B \subseteq X$. Then $\mu(B \cap A) = 0$, and consequently

$$\mu(B) \geq \mu(B - A) = \mu(B \cap A) + \mu(B - A).$$

The opposite inequality is clear from subadditivity.

3. Assume A is μ-measurable, $B \subseteq X$. Then

$$\begin{aligned} \mu \llcorner C(B) &= \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) - A) \\ &= \mu((B \cap A) \cap C) + \mu((B - A) \cap C) \\ &= \mu \llcorner C(B \cap A) + \mu \llcorner C(B - A). \end{aligned}$$

Hence A is $\mu \llcorner C$-measurable. □

THEOREM 1.2 (Sequences of measurable sets). *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of μ-measurable sets.*

(i) *The sets*

$$\bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k$$

are μ-measurable.

(ii) *If the sets $\{A_k\}_{k=1}^{\infty}$ are disjoint, then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

(iii) *If $A_1 \subseteq \ldots A_k \subseteq A_{k+1} \ldots$, then*

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

(iv) *If $A_1 \supset \ldots A_k \supset A_{k+1} \ldots$ and $\mu(A_1) < \infty$, then*

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right).$$